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The Jacobi product formula

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Theorem: *(Jacobi Product Formula)* Let w, q be indeterminates. Then in the formal power series ring $\mathbf{Z}[w, w^{-1}][[q]]$ in q over the ring $\mathbf{Z}[w, w^{-1}]$

$$1 + \sum_{n \geq 1} (w^{2n} + w^{-2n})\, q^{n^2} = \prod_{n \geq 1} (1 - q^{2n})\, (1 + q^{2n-1}\, w^2)\, (1 + q^{2n-1}\, w^{-2})$$

Corollary: For $q \in \mathbf{C}$ and $w \in \mathbf{C}$ with $|q| < 1$ the Jacobi Product Formula holds as an equality of holomorphic functions. For example, for z in the complex upper half-plane, letting $q = e^{\pi i z}$,

$$\sum_{n \in \mathbf{Z}} e^{\pi i n^2 z} = \prod_{n \geq 1} (1 - q^{2n})\, (1 + q^{2n-1})\, (1 + q^{2n-1})$$

In particular, this expression is non-zero for z in the complex upper half-plane. ♣

Proof: Consider the partial product

$$\varphi_m(q, w) = \prod_{1 \leq n \leq m} (1 + q^{2n-1} w^2)\, (1 + q^{2n-1} w^{-2})$$

Regroup the terms of $\varphi_m(q, w)$ by powers of w^2 and w^{-2}:

$$\varphi_m(q, w) = \Phi_{m,0} + (w^2 + w^{-2})\, \Phi_{m,1} + \ldots + (w^{2m} + w^{-2m})\, \Phi_{m,m}$$

for suitable polynomials $\Phi_{m,i} \in \mathbf{Z}[q]$. The top polynomial $\Phi_{m,m}$ is easy to determine, namely

$$\Phi_{m,m} = q^{1+3+5+\ldots+(2m-3)+(2m-1)} = q^{m^2}$$

We will obtain a recursion for the other $\Phi_{m,n}$'s. Replacing w by qw in φ gives

$$\varphi_m(q, qw) = \prod_{1 \leq n \leq m} (1 + q^{2n+1} w^2)\, (1 + q^{2n-3} w^{-2})$$

$$= (1 + q^{2m+1} w^2)\, (1 + q^{-1} w^{-2})\, (1 + q w^2)^{-1}\, (1 + q^{2n-1} w^{-2})^{-1}\, \varphi_m(q, w)$$

just by regrouping factors. The middle two leftover factors nearly cancel, leaving

$$(q w^2 + q^{2m})\, \varphi_m(q, qw) = (1 + q^{2m+1} w^2)\, \varphi_m(q, w)$$

Equating the coefficients of w^{2-2n} in the latter equality yields (for $n \geq 1$)

$$q \cdot \Phi_{m,n} \cdot q^{-2n} + q^{2m} \cdot \Phi_{m,n-1} \cdot q^{2-2n} = \Phi_{m,n-1} + q^{2m+1}\, \Phi_{m,n}$$

from which we obtain

$$\Phi_{m,n} = q^{2n-1}\, (1 - q^{2m-2n+2})\, (1 - q^{2m+2n})^{-1}\, \Phi_{m,n-1}$$

Thus, by the obvious induction we obtain the product expansion

$$\Phi_{m,n} = q^{n^2} \left(\prod_{1 \leq i \leq n} \frac{1 - q^{2m-2i+2}}{1 - q^{2m+2i}} \right) \Phi_{m,0}$$

We already have $\Phi_{m,m} = q^{m^2}$, so

$$q^{m^2} = \Phi_{m,m} = q^{m^2} \left(\prod_{1 \le i \le m} \frac{1 - q^{2m-2i+2}}{1 - q^{2m+2i}} \right) \Phi_{m,0}$$

from which we conclude that

$$\Phi_{m,0} = \prod_{1 \le i \le m} \frac{1 - q^{2m+2i}}{1 - q^{2m-2i+2}} = \prod_{1 \le i \le m} \frac{1 - q^{2m+2i}}{1 - q^{2i}}$$

by replacing i by $m + 1 - i$ in the denominators (and rearranging). Then

$$\Phi_{m,n} = q^{n^2} \left(\prod_{1 \le i \le n} \frac{1 - q^{2m-2i+2}}{1 - q^{2m+2i}} \right) \left(\prod_{1 \le i \le m} \frac{1 - q^{2m+2i}}{1 - q^{2i}} \right)$$

As $i \to \infty$, $q^i \to 0$, by definition of the topology on $\mathbf{Z}[[q]]$. Thus, for fixed n, as $m \to \infty$, except for the factors $1 - q^{2i}$, the factors in $\Phi_{m,n}$ go to 1 in $\mathbf{Z}[[q]]$. That is, for any fixed $n \ge 1$, as $m \to \infty$

$$\Phi_{m,n} \to \prod_{1 \le i} \frac{1}{1 - q^{2i}}$$

Thus, the equality

$$\prod_{1 \le n \le m} (1 + q^{2n-1} w^2)\,(1 + q^{2n-1} w^{-2}) = \Phi_{m,0} + \sum_{1 \le n \le m} q^{n^2}\,(w^{2n} + w^{-2n})\,\Phi_{m,n}$$

for each fixed m (by which the $\Phi_{m,n}$ were initially defined) becomes, as $m \to \infty$,

$$\prod_{1 \le n} (1 + q^{2n-1} w^2)\,(1 + q^{2n-1} w^{-2}) = \left(\prod_{1 \le i} \frac{1}{1 - q^{2i}} \right) \left(1 + \sum_{1 \le n} q^{n^2}\,(w^{2n} + w^{-2n}) \right)$$

which gives

$$\prod_{1 \le n} (1 - q^{2n})\,(1 + q^{2n-1} w^2)\,(1 + q^{2n-1} w^{-2}) = 1 + \sum_{1 \le n} q^{n^2}\,(w^{2n} + w^{-2n})$$

which is the asserted identity. ♣