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Geometric homology versus group homology

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The goal here is to recover a family of assertions of the form

$$H^*(\Gamma\backslash X) \approx H^*(\Gamma)$$

where the left-hand side is *singular* cohomology of a topological space $\Gamma\backslash X$ obtained as a quotient of a homeomorphic copy of an open ball X by the action of a discrete group Γ, and the right-hand side is *group* cohomology of Γ. A striking aspect of this is that the essence of the argument concerns *homotopy* rather than *(co-) homology*.

A brief outline of the early history, as in [Dieudonné 1989] and [Maclane 1963], is as follows. (See also [Weibel].) The fact that homology (or at least the collection of Betti numbers) is determined by $\pi_1(X)$ for X having $\pi_i(X) = \{1\}$ for $i > 1$ was known from [Hurewicz 1935]. Nascent ideas concerning tangible applications of low-degree group (co-) homology appeared in [Baer 1934]. But explicit formation of the notion of group cohomology (beyond H^1 and H^2, and perhaps H^3) as an artifact of a construction of spaces with specified π_1 and no higher homotopy occurred in [Eilenberg MacLane 1943] and [Eilenberg MacLane 1945], and independently in [Eckmann 1946]. The homology assertion was in [Hopf 1945] and independently in [Freudenthal 1946]. [MacLane 1963] also cites [Hopf 1942] as a predecessor, treating the second Betti group, but this source is less widely available.

The functorial construction of a CW-complex $K(G, 1)$ given here roughly follows the readable account in [Hatcher 2002]. It is a relatively simple matter to observe that the construction does incidentally yield the (by-now standard!) **bar construction** of a free resolution of the trivial G-module. Once this is realized, the isomorphism of the singular and algebraic homology is immediate.

For those who have wondered about the motivation for group (co-) homology beyond H^1, H^2, and perhaps H^3, this history may indicate that discovery of (co-) homology *functorially* attached to a group G was an accidental side-effect of a *functorial* construction of the corresponding $K(G, 1)$[1] by Eilenberg and MacLane. Hurewicz studied the $n = 1$ case of the topological problem. Eilenberg and MacLane treated the case of general n, and observed that the homotopy invariance of (co-) homology groups attached to a *functorially* constructed $K(G, 1)$ gave a (co-) homology theory for groups as a by-product. Indeed, the so-called *bar resolution* for group (co-) homology is the construction of $K(G, 1)$ with the geometry *removed*. Further, the fact that the (co-) homology groups of a $K(G, 1)$ are independent of constructions (via homotopies) apparently suggested that group (co-) homology should not depend upon the specific resolution. Thus, the mystery of the motivation for higher group (co-) homology is partly resolved by understanding it as an artifact of the construction of $K(G, 1)$'s and $K(G, n)$'s.

However, even if we take this somewhat skeptical view of the creation or discovery of group (co-) homology, by now we appreciate it as consisting of derived functors of the functors that take co-fixed (resp. fixed) vectors in a representation. That is, (co-) homology spaces have a natural existence, so, as usual, it would be unwise to ignore them, regardless of history.

[1] As below, a topological space X is a $K(G, 1)$ if it has $\pi_1(X) \approx G$ and $\pi_i(X) = \{1\}$ for $i > 1$. Sometimes one requires that X be a CW-complex. The functorial construction here rarely yields a finite-dimensional CW-complex, so accommodation must be made for the possibility of infinite-dimensional spaces. Similarly, a $K(G, n)$ has prescribed $\pi_n(X) \approx G$ and other homotopy trivial.

- CW-complexes and Δ-complexes
- A functorial construction of $K(G,1)$
- Homotopy type of $K(G,1)$'s
- Homotopy of contractible universal covers
- Spaces $\Gamma\backslash X$ with contractible X
- Appendix: contractibility of spheres
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We will prove that the homotopy type of a connected CW-complex $X = K(G,1)$ (with $\pi_1(X) = G$ and vanishing higher homotopy) is uniquely determined by the group G, and then *construct* a CW-complex [2] $BG \simeq K(G,1)$ whose singular chain complex is literally the bar resolution of G in group homology. Finally, we discuss the simple observation that this construction of BG gives a free resolution of the trivial G-module, thus yielding group homology.

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1. CW-complexes

A **CW-complex**[3] is a nice topological space created by sticking together balls of various dimensions. [4]

More precisely, let B_n be the closed unit ball in $\mathbf{R}^n$, and let $S^{n-1} = \partial B_n$ be the $(n-1)$-sphere bounding B_n. An n-**cell** is a topological space e^n homeomorphic to B_n. Collections of n-cells for $n = 0, 1, 2, \ldots$ are assembled into a CW-complex X as follows.

We begin by specifying a discretely topologized set X^0, the 0-skeleton of X.

Inductively, granting that the $(n-1)$-skeleton X^{n-1} of the CW-complex X is already constructed, we specify a set of n-cells e_i^n and **attaching maps**

$$\varphi_i^n : \partial e_i^n \approx S^{n-1} \longrightarrow X^{n-1}$$

Then the n-skeleton X^n is the quotient

$$X^n = \left(X^{n-1} \sqcup \bigsqcup_i e_i^n \right) / \sim$$

where $\sim$ is the equivalence relation which identifies each $x \in \partial e_i^n$ with its image $\varphi_i^n(x) \in X^{n-1}$. By abuse of language, we identify each cell e_i^n with its image in X^n.

The whole CW-complex $X = \bigcup_n X^n$ is the colimit (in the category of topological spaces), and thus has a uniquely specified *colimit topology*. That is, X has the unique topology [5] such that every inclusion $j_n : X^n \longrightarrow X$ is continuous, and such that for every family of maps

$$f_n : X^n \longrightarrow Y$$

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[2] The notation BG for this construction of a functorial $K(G,1)$ is standard.

[3] As in [Hatcher 2002], the C refers to *closure finiteness*, meaning that the closure of each cell meets only finitely-many other cells, and the W refers to the topology, sometimes called the *weak* topology. (Since it is literally a colimit topology, we will call it the *colimit topology*.)

[4] Although there are technical reasons to discuss *simplices* and *simplicial complexes* prior to discussion of CW-complexes, in the end the latter viewpoint is much more convenient, if only as a more flexible descriptive language.

[5] More important than the name *per se*, the acknowledgement that the topology is that of a colimit tells quite explicitly why we take this topology, since we make clear its mapping properties in advance.

with the compatibility condition that for $m < n$ the diagram

$$\begin{tikzcd} X_m \arrow[rr, "\subset"] \arrow[dr, "f_m"'] & & X_n \arrow[dl, "f_n"] \\ & Y & \end{tikzcd}$$

commutes, there is a unique $f : X \longrightarrow Y$ such that all diagrams

$$\begin{tikzcd} X_n \arrow[rr, "j_n"] \arrow[dr, "f_n"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

commute. This universal mapping property characterization proves the uniqueness up to unique isomorphism. Existence of the colimit is a little boring here, because the maps among the X_n are so simple. That is, take the quotient of the coproduct (disjoint union) $\bigsqcup_n X^n$ by identifying $x_m \in X_m$ with its image in X_n for $m < n$, via $X_m \subset X_n$. It is not so hard to verify that this gives a coproduct whose underlying set is the ascending union (which is itself a set-coproduct).

A very special kind of CW-complex construction which is still much more flexible than simplicial complex constructions is what [Hatcher 2002] calls **Δ-complexes**, which are used to construct $K(G,1)$'s functorially in a fashion which incidentally creates the bar resolution for group homology.

As usual, a **(geometric) n-simplex** is (a topological space σ homeomorphic to) the convex hull in some Euclidean space $\mathbf{R}^N$ of $n+1$ affinely-independent[6] points $v_0, \ldots, v_n$.[7] The **vertices** of σ are the points x_i of which σ is the convex hull. An **orientation** of σ is a choice of ordering of its vertices. We will often name an n-simplex by listing its vertices v_i in the ordering given by the orientation, in the notation

$$\sigma = [v_0, v_1, \ldots, v_n]$$

The i^{th}**-codimension-one face** of σ is the convex hull of

$$x_0, x_1, \ldots, x_{i-1}, x_{i+1}, x_n$$

That is, the i^{th} codimension-one face of $\sigma = [x_0, \ldots, x_n]$ is

$$[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = [x_0, \ldots, \widehat{x}_i, \ldots, x_n]$$

where as usual the hat denotes omission. More generally, given a subset $I = \{i_0.i_1, \ldots, i_d\}$ of $\{0, 1, \ldots, n\}$ with

$$i_0 < \ldots < i_d$$

the I^{th} face of $\sigma = [x_0, x_1, \ldots, x_n]$ is

$$[x_{i_0}, x_{i_1}, \ldots, x_{i_d}]$$

In particular, the faces of a simplex σ inherit an orientation from σ.

Unlike the more restrictive rules for constructing simplicial complexes, we construct Δ-complexes as follows. Starting with a disjoint union of simplices, we are allowed to *identify* two faces F and F' (of the same

[6] That is, there is no relation $\sum_i t_i v_i = 0$ for real numbers $t_0, \ldots, t_n$ with $\sum_i t_i = 1$.

[7] The dimension N of the ambient Euclidean is allowed to be much larger than the number $n+1$ of points. If one resists identifying simplices σ with subsets of Euclidean spaces, then one needs to specify homeomorphisms φ_σ from the convex hulls to σ, and so on.

dimension d) of simplices σ and σ' in a restricted fashion. We view F and F' as sitting in different ambient Euclidean spaces, and let

$$L : (\text{affine span of } x_0, \ldots, x_d) \longrightarrow (\text{affine span of } y_0, \ldots, y_d)$$

be the unique (thus, canonical) linear map (on the linear span of the x_i) such that

$$L(x_i) = y_i$$

for all i. Then we are allowed to identify F and F' via L. This does make the orientations of F and F' match, and gives compatible identifications of the faces of the simplices F and F' by continuity (indeed, by linearity).[8]

We have the expected

Proposition: A Δ-complex X is a CW-complex.

Proof: We follow the appendix of [Hatcher 2002]. This argument amounts to nearly trivial modifications to match the formal description of CW-complexes. The fact itself is quite believable at the outset.

The interior σ^o of the convex hull σ of $n+1$ points in $\mathbf{R}^n$ is an open n-ball e^n_σ. The boundary of σ is homeomorphic to an $n-1$ sphere S^{n-1}, and the induced map of S^{n-1} to the $(n-1)$-skeleton X^{n-1} of X is plausibly continuous. With a little modification for formalities, any Δ-complex is a CW-complex made from these cells and the obvious attaching maps.

The modifications are as follows. Given a Δ-complex X, enlarge the set of simplices and identifications by including copies of all the lower-dimensional faces of the simplices σ in X, with identifications (linear maps rigidly constrained as above, in the definitions) to the corresponding faces of σ. (If necessary, remove duplicates of simplices which are entirely identified in X.) Then the given Δ-complex X is the (disjoint union of) cells e^n_σ modulo implied attaching maps.[9]

We prove by induction that this presents X as a CW-complex, while explicating the attaching maps. Let X^n be the union of all the m-cells with $m \le n$, glued together by the corresponding attaching maps, and assume that the $(n-1)$-skeleton X^{n-1} is a CW-complex (with implied attaching maps). This is true for $n-1=0$, since a 0-dimensional CW-complex has no attaching maps, being a disjoint union of points. By induction, for each cell $e^m_\sigma = \sigma^o$ with $m \le n-1$, the restriction

$$\sigma^o = e^m_\sigma \longrightarrow X^{n-1}$$

of the continuous map $\sigma \longrightarrow X$ to the interior σ^o of σ is continuous. The boundary of σ is homeomorphic to S^{n-1}, and the lower-dimensional faces of σ have already been assembled in X^{n-1} into an image of this sphere. Thus, for each n-cell σ, we have a genuine attaching map

$$\varphi_\sigma : S^{n-1} \longrightarrow X^{n-1} \subset X^n$$

Thus, X^n is also a CW-complex.

By definition, a colimit (with inclusions) of CW-complexes (with the inclusions arranged as attaching maps, as we have) is a CW-complex. ///

[8] Thus, in a Δ-complex, unlike in a simplicial complex, it is not true that the vertices of a simplex determine the simplex uniquely. Tolerating this allows constructions using far fewer simplices than would be required by simplicial complexes. This is an advantage enjoyed more broadly by CW-complexes. At the same time, the Δ-complex construction is far more rigid (and completely combinatorial) than the general CW-complex.

[9] It is not completely proper to call these attaching maps before we know that the whole is a proper CW-complex, but we do anticipate that they are.

2. A functorial construction of $K(G,1)$

The usual definition of a $K(G,n)$ is that it is a (connected) topological space X with prescribed n^{th} homotopy group $\pi_n(X) \approx G$, and all other homotopy groups trivial. The $K(G,1)$'s are also called *classifying spaces*, and often denoted as BG.

The construction of $K(G,1)$'s here is the **bar construction** or **bar resolution**. These While this construction creates large spaces (demonstrably larger than necessary in some simple cases), it has the virtue of functoriality.

Further, with some hindsight (which we'll acquire just below) it will be seen that the algebra of this construction gives a free resolution of the trivial G-module, if one simply ignores the geometry. That is, an idiosyncratic geometrically motivated construction, when the geometry is removed, yields group cohomology. This peculiar causal relationship will be clarified below.

In the standard notation, given a group G, let EG be the Δ-complex (as above) whose n-simplices are all ordered $(n+1)$-tuples

$$\sigma = [g_0, \ldots, g_n]$$

of elements of G, with natural identifications of faces

$$[g_0, \ldots, \widehat{g}_i, \ldots g_n]$$

A critical observation is that EG is *contractible.* [10] Indeed, let h be the homotopy that slides each point x in $[g_0, \ldots, g_n]$ along the line segment in the $(n+1)$-simplex $[e, g_0, \ldots, g_n]$ (with e the identity in G) from x to the 0-simplex (vertex) $[e]$. Since there are many implied identifications in EG, one should stop to be sure that this homotopy h is well-defined. Specifically, the homotopy of $[g_0, \ldots, g_n]$ inside $[e, g_0, \ldots, g_n]$ does restrict to the corresponding homotopy in $[e, g_0, \ldots, \widehat{g}_i, \ldots, g_n]$ on a face $[g_0, \ldots, \widehat{g}_i, \ldots, g_n]$. [11]

The group G acts on EG on the left by Δ-complex maps[12] by affine maps, specified on the vertices in the natural fashion

$$g \cdot [g_0, \ldots, g_n] = [gg_0, \ldots, gg_n]$$

and extending to the convex hull by requiring affine-ness. We claim that the map $EG \longrightarrow G\backslash EG$ to the quotient $BG = G\backslash EG$ of EG by the G-action is a *covering space*. Since EG is contractible, this map $EG \longrightarrow BG$ exhibits EG as a *universal* covering space of BG, with covering group G. Thus, $\pi_1(BG) \approx G$, by covering-space theory (as in the appendix). ///

Remark: Since G acts *properly*[13] on EG by permuting simplices, the quotient BG inherits a natural Δ-complex structure, in which there is a single vertex. Writing a simplex in EG in the style

$$[g_0,\ g_0g_1,\ g_0g_1g_2,\ \ldots,\ g_0g_1\cdots g_n] = g_0[e,\ g_1,\ g_1g_2,\ \ldots,\ g_1\cdots g_n]$$

[10] Recall that contractibility implies that the identity is homotopic to a constant map. This should *not* be misconstrued as reliably suggesting that the space has a retraction to a point.

[11] As [Hatcher 2002] notes, this homotopy is not a retraction to $[e]$. Indeed, the 1-simplex (edge) $[e,e]$ has two faces, $[e]$ and $[e]$, which are the same point. Since this is a Δ-complex and not a simplicial complex, this is permissible, and we find that $[e,e]$ is homeomorphic to a circle. The same is true of any $[g,g]$ with $g \in G$. Thus, the homotopy moves $[e]$ along the *loop* $[e,e]$.

[12] The precise definition of Δ-complex map is left as an exercise.

[13] The properness of the action is the visible property that no element of G other than the identity fixes any point of EG.

we finally introduce the **bar notation**

$$[g_1|\ldots|g_n] = \text{image in } G\backslash EG \text{ of } [e,\ g_1,\ g_1g_2,\ \ldots,\ g_1\ldots g_n]$$

Then in BG the topological boundary of $[g_1|\ldots|g_n]$ is union of simplices

$$[g_1|\ldots|g_ig_{i+1}|\ldots|g_n] \quad (\text{for } 1 \le i \le n)$$

and also the *end cases*, two more simplices,

$$[g_2|\ldots|g_n] \quad \text{and} \quad [g_1|\ldots|g_{n-1}]$$

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3. Homotopy type of $K(G,1)$'s

Examples can show that there is a considerable variety of constructions of spaces as $K(G,1)$s, but, nevertheless,

Theorem: The homotopy type of a connected CW-complex $K(G,1)$ is completely determined by G.

Corollary: Any homotopy-invariant functor F on topological spaces yields a functor on groups G by $G \longrightarrow K(G,1) \longrightarrow FK(G,1)$. ///

Proof: Grant for the moment that for a CW-complex X and a CW-complex $Y = K(G,1)$ that every group homomorphism

$$\pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0) \quad (\text{with } x_0 \in X,\ y_0 \in Y)$$

is induced by a map $X \longrightarrow Y$ sending $x_0 \longrightarrow y_0$ that is unique up to a homotopy fixing x_0. Then for X *another* $K(G,1)$, an isomorphism

$$\pi_1(X,x_0) \approx G \approx \pi(Y,y_0)$$

yields maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ sending $x_0 \leftrightarrow y_0$, and unique up to homotopy. Thus, $f \circ g$ and $g \circ f$ induce the identity maps on the π_1s, and such maps are unique up to homotopy. The identity maps on X and Y also induce the identity maps on the π_1s, so, by the uniqueness up to homotopy, $f \circ g$ and $g \circ f$ are homotopic to the identity.

Now we prove that every group homomorphism $\pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$ with $x_0 \in X,\ y_0 \in Y$ is induced by a map $X \longrightarrow Y$ sending $x_0 \longrightarrow y_0$, unique up to a homotopy fixing x_0. To this end, first consider the simple case that X is a CW-complex with a single vertex x_0. [14] Let $f : \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$ be a homomorphism. We will create a continuous map $F : (X,x_0) \longrightarrow (Y,y_0)$ so that the induced map F_* on π_1 is $F_* = f$. First, of course, define $F(x_0) = y_0$. Second, each 1-cell e^1 of X has closure $e^1 \cup x_0 \approx S^1$, which gives a class we'll denote $[e^1]$ in the 1-skeleton's $\pi_1(X^1,x_0)$. Define F on the closure $e^1 \cup x_0 \subset X^1$ so that $F_*[e^1] = f[e^1]$. This gives an extension of F to the 1-skeleton X^1 of X.

Given an attaching map $\alpha : S^1 \longrightarrow X^1 \subset X$ in the CW-complex structure of X, to extend F to the 2-cell e^2 attached along α, it suffices that $F \circ \alpha$ is nullhomotopic in Y: given a nullhomotopy $h_t : S^1 \times [0,1] \longrightarrow Y$ with $h_0(S^1)$ a fixed point $h_0(S^1) = s_0$, observe that h_0 factors through the *cone* on S^1, which is a disc. Thus, this e^2 is attached along $F \circ \alpha$ by a nullhomotopy in Y of $F \circ \alpha : S^1 \longrightarrow Y$, as claimed.

To find a nullhomotopy in Y, let $j : X^1 \longrightarrow X$ be the inclusion, let s_0 be a basepoint on S^1, and fix a path in X^1 from $\alpha(s_0)$ to x_0 (using the fact that X^1 has a single vertex). With the choices of base point and

[14] Since the bar construction produces CW-complexes with a single vertex, this part of the argument would already suffice.

path from x_0 to it, α specifies a class we'll denote $[\alpha]$ in $\pi_1(X^1, x_0)$. Existence of a homotopy of $F \circ \alpha$ to a constant map in Y is (of course) equivalent to $F_* \circ j_*([\alpha])$ being 1 in $\pi_1(Y, y_0)$. Already $j_*([\alpha])$ is 1 in $\pi_1(X, x_0)$, so its image under the group homomorphism f is certainly 1 in $\pi_1(Y, y_0)$.

To extend F to cells of dimensions $n > 2$ in X, we again use the fact that a nullhomotopy of a map of S^{n-1} to a space factors through the cone of S^{n-1}, which is a closed n-ball. Thus, we claim that every attaching map $\alpha : e^n \longrightarrow X^{n-1}$ give $F \circ \alpha : S^{n-1} \longrightarrow Y$ which is nullhomotopic. Indeed, since $n > 2$, the map $F \circ \alpha$ lifts to a map Φ to the universal covering of Y [15] This universal covering is simply-connected, so Φ is nullhomotopic, and the nullhomotopy descends to a nullhomotopy in Y. By its definition, the colimit topology on X gives a continuous $F : (X, x_0) \longrightarrow (Y, y_0)$ inducing the given group homomorphism f on π_1.

For uniqueness (up to homotopy) of F, let Φ be another map inducing the same homomorphism on π_1. Restricted to the 1-dimensional CW-complex X^1 (with a single vertex!), the fact that $\Phi_* = F_*$ says that for any attaching map $\alpha : \{0, 1\} \longrightarrow \{x_0\}$ and corresponding attachment of an e^1 to make $e^1 \cup \{x_0\} \approx S^1$, the images $F(e^1 \cup \{x_0\}$ and $\Phi(e^1 \cup \{x_0\})$ are homotopic in Y, by homotopies fixing $y_0 = F(x_0) = \Phi(x_0)$. All these separate homotopies can be assembled to give a homotopy h between the restrictions to X^1 of F and Φ.

To extend the homotopy

$$h : X^1 \times [0, 1] \longrightarrow Y$$

(from F to Φ on X^1) we first extend h to

$$H : X^1 \times (0, 1) \cup X \times \partial[0, 1] \longrightarrow Y$$

where on $X \times \{0\}$ h' is F, while on $X \times \{1\}$ h' is Φ. Note that the remaining cells $e^n \times (0, 1)$ of the CW-complex $X \times [0, 1]$ have dimension $n + 1 > 1 + 1 = 2$. Thus, the extension argument above that proved *existence* proves the extendability of H to a homotopy of F to Φ. This proves existence and uniqueness in the case that X has a single vertex.

The case that X has more than a single vertex is a version of the ideas of the single-vertex case but with a slightly more complicated beginning, as follows. Let T be a *maximal tree*[16] in X^1. To construct the desired F, first set $F(T) = y_0$. Next, by the maximality of T, an edge e^1 in $X - T$ gives a homotopy class we'll denote $[e^1]$. Define F on the closure of e^1 to be any map giving $f[e^1]$ in homotopy. The construction of F on higher-dimensional cells is as before, proving existence.

To prove uniqueness (up to homotopy) in the general case, again let T be a maximal tree in X^1, and begin with a deformation retraction r of T to x_0, which exists since T is contractible to x_0. We claim first that r extends to a homotopy h from the identity $X^1 \longrightarrow X^1$ to some other map on X^1. The extension of r will not move vertices e^0 not in T, and will not move edges e^1 attached to two vertices both not in T. Thus, to define the extension we need only consider edges e^1 attached to at least one vertex *in* T, and attached to another vertex which may or may not be in T. We can treat both cases simultaneously by deciding in either case to keep (in effect) the *midpoint* of e^1 *fixed*. The half of e^1 attached to a vertex *not* in T (if any) will not move at all. Meanwhile, any half of e^1 attached to a vertex e^0 in T should *stretch* to follow e^0 as it moves toward x_0 under the retraction r.

Given a homotopy h on X^1 which restricts to the retraction of T to x_0, and given two maps F and Φ inducing the given homomorphism on π_1, a homotopy from F on X^1 to Φ on X^1 can be constructed by composing F and Φ (restricted to X^1) with h. This reduces to the single-vertex case for X^1 with these modifed F and Φ. Then the remainder of the earlier argument proceeds in the same fashion, once we have the homotopy on X^1. ///

[15] See the two appendices, one on contractibility of spheres, the other on the homotopy criterion for lifting of maps.

[16] As usual, a *tree* is a connected 1-dimensional simplicial complex with no *cycles*, that is, with trivial π_1.

4. Homotopy of contractible universal covers

As reviewed in [Rosenberg 1994], among other results [Dold 1963] includes a very useful result on the homotopy type of quotients $\Gamma\backslash X$ where X is contractible and Γ acts nicely. We reproduce the result and Dold's argument below. This result will allow us to apply results on CW-complex $K(G,1)$'s to quotients of symmetric spaces X [17] by arithmetic subgroups Γ, and then convert (co-) homological questions about these quotients $\Gamma\backslash X$ to questions about the group (co-) homology of Γ itself.

Let $p: E \longrightarrow B$ be a continuous map. [18] As usual, a continuous map $s: B \longrightarrow E$ is a **section** for p if it is a one-sided inverse, namely, if

$$p \circ s = 1_S$$

In Dold's terminology, for $A \subset B$, a **halo** around A is a set H with $A \subset H \subset B$ with a continuous $[0,1]$-valued function f (a *halo function*) on all of B such that $f = 1$ on A and $f = 0$ off H. For a subset A of B, let $E_A = p^{-1}A$ and

$$p_A : E_A \longrightarrow A$$

be the corresponding restriction of $p: E \longrightarrow B$. The map $p: E \longrightarrow B$ has the **section extension property** if, for all $A \subset B$ and for all sections s of p_A extending to a section of p_H for some halo H of A,[19] *s extends* to a section $\sigma : B \longrightarrow E$. [20] Here s is called a **local section** and σ a **global section**.

Lemma: A map $p: E \longrightarrow B$ has the section extension property for $E = B \times X$ for a contractible space X and p the projection to B.

Proof: Let $h : X \times [0,1] \longrightarrow X$ be a contraction, that is, a continuous map such that $h(x,0) = x$ and $h(x,1) = x_0$ for some x_0, for all $x \in X$. Let H be a halo around $A \subset B$, with halo function f. Let s be a section of p_A extending to a section s' of p_H. Let $q : E = B \times X \longrightarrow X$ be the projection to the second factor. Define a section σ of p by[21]

$$\sigma(b) = \begin{cases} b \times x_0 & (\text{for } b \notin H) \\ b \times h(q \circ s'(b), f(b)) & (\text{for } b \in H) \end{cases}$$

This extends the section s. [22] ///

Lemma: Let $p : E \longrightarrow B$ have the section extension property. For continuous $\varphi : B \longrightarrow [0,1]$, the restriction $p_U : E_U \longrightarrow U$ to the inverse image under p of the open set

$$U = \varphi^{-1}(0,1] \subset B$$

[17] For present purposes, we are only interested in symmetric spaces which happen to be contractible. Typically these would be called symmetric spaces *of non-compact type*, although this terminology does not reveal that they are contractible. The complex upper half-plan, complex n-balls, real n-balls, and Siegel upper half-planes are popular examples of non-compact type symmetric spaces.

[18] Soon $p: E \longrightarrow B$ will be a covering map, with p for *projection* and B for *base*.

[19] The extendability of the section to some *halo* of A is an adroit general means to exclude several pathologies.

[20] There is no claim that the extension of the section to the halo extends to a section on all of B.

[21] The trick is that this section uses the halo function as the $[0,1]$-parameter in the contraction homotopy.

[22] Note that the section extension makes use of the extension to the halo, but does not extend this extension itself.

also has the section extension property.

Proof: *(Dold)* Let $f : U \longrightarrow [0,1]$ be continuous, put $A = f^{-1}(1)$, and let s be a section of p over $f^{-1}(0,1]$. [23] The latter is open in U, and, thus, open in B. To prove the section extension property, we need a section σ of p_U agreeing with s restricted to A. Dold inductively constructs sections s_n of p on $\varphi^{-1}(\frac{1}{n},1]$ (agreeing on overlaps) converging to σ. The convergence to σ is assured by requiring at each step that

$$s_n = s \quad \text{on} \quad \varphi^{-1}(\frac{1}{n+1},1] \cap f^{-1}(\frac{n-1}{n},1]$$

which implies that eventually $s_n(b) = s(b)$ for $b \in U \cap f^{-1}(1)$. To start, we want a section agreeing with s on the set S where $\varphi > \frac{1}{3}$ and $f > \frac{1}{2}$. To this end, note that the product $f\varphi$ extends to be continuous on all of B by defining it to be 0 off U. Then s extends from S to the corresponding *halo*

$$H = \{b \in B : (f\varphi)(b) > 0\}$$

Thus, s has a global extension s_2 by the section extension property for p. Now assume we have $s_2, \ldots, s_n$, and construct s_{n+1} agreeing with s_n on $\varphi^{-1}(\frac{1}{n},1]$ and agreeing with s on

$$\varphi^{-1}(\frac{1}{n+2},1] \cap f^{-1}(\frac{n}{n+1},1]$$

To this end, let

$$k_n : [0,1] \longrightarrow [\frac{1}{n+2},\frac{1}{n}]$$

be continuous and decreasing, requiring further that

$$k_n(t) = \begin{cases} \frac{1}{n} & (\text{for } t \le \frac{n-1}{n}) \\ \frac{1}{n+2} & (\text{for } t \ge \frac{n}{n+1}) \end{cases}$$

Let

$$A_n = \{b \in U : \varphi(b) > k_n(f(b))\}$$

Since s_n and s agree on the set

$$f > \frac{1}{n+1} \quad \text{and} \quad g > \frac{n-1}{n}$$

we can define a section σ of p over A_n by

$$\sigma(b) = \begin{cases} s_n(b) & (\text{for } f(b) > \frac{1}{n+1}) \\ s(b) & (\text{for } g(b) > \frac{n-1}{n}) \end{cases}$$

At least one of these conditions holds at every point of A_n. We must extend this section to a section on a halo of A_n and then invoke the section extension property for p. We claim that

$$H_n = \{b \in U : \varphi(b) > \frac{n}{n+1} \cdot k_n(f(b))\}$$

is a halo of A_n. Indeed, we can define a halo function for H_n by

$$h_n(b) = \begin{cases} 0 & (\text{for } b \notin H_n) \\ 1 & (\text{for } b \in A_n) \\ \frac{(n+1)\varphi(b) - nk_n(f(b))}{k_n(f(b))} & (\text{for } b \in H_n - A_n) \end{cases}$$

[23] We try to suppress parentheses when possible. As usual, we claim that context does disambiguate the varying usage.

This function h_n is continuous, exhibits H_n as a halo of A_n, so a global section s_{n+1} exists, by the section extension property of p. ///

Theorem: *(Dold)* Let B be *paracompact*,[24] E and X Hausdorff, and $p : E \longrightarrow B$ an *open* continuous map such that each $b \in B$ has a neighborhood U_b such that there is a homeomorphism

$$\varphi_b : p^{-1}(U_b) \stackrel{\simeq}{\longrightarrow} U_b \times X$$

with

$$p|_{p^{-1}(U_b)} = \mathrm{pr}_{U_b} \circ \varphi_b$$

where $\mathrm{pr}_{U_b} : U_b \times X \longrightarrow U_b$ is the projection on the first factor. Let X be *contractible*. Then p is a *homotopy equivalence*. Further, there is a section s of p such that there is a *vertical* homotopy from 1_E to $s \circ p$, meaning that the homotopy commutes with p.[25]

Proof: It suffices to show that p has the section extension property and then take the set A to be the empty set. Since B is paracompact, the covering $\{U_b : b \in B\}$ admits a locally finite refinement $\{V_i : i \in I\}$ for some index set I. Let $\{f_i : i \in I\}$ be a partition of unity subordinate to this locally finite refinement.[26] By the first lemma, each p_{V_i} has the section extension property, and by the second lemma this will still be the case upon replacing V_i by $f_i^{-1}(0,1]$. Let $A \subset B$ and $A = f^{-1}(1)$ for a continuous $[0,1]$-valued function f on B. Let s be a section of p_V, where $V = f^{-1}(0,1]$. We must show that $s|_A$ has a an extension to a global section of p.

Now comes a surprising part. For a subset J of the index set I, let

$$V_J = \bigcup_{i \in J} V_i$$

Let S be the set of pairs (J, s_J) where J is a subset of the index set I and s_J is a section of $p_{V \cup V_J}$ extending $s|_A$. Order S by

$$(J, s_J) \leq (J', s_{J'}) \quad \text{if} \quad J \subset J' \ \text{ and } s_{J'} \text{ extends } s_J$$

Since it contains (ϕ, s), the set S is non-empty. Every *chain* in S visibly has an upper bound, so by Zorn's Lemma S has a maximal element (K, s_K). It is natural to claim that $V \cup V_K = B$. If this were not so, there would be an index j with V_j not entirely contained in $V \cup V_K$. For this index j, define

$$h(b) = \min\,(1,\ \frac{f(b) + \sum_{i \in K}\ f_i(b)}{f_j(b)})$$

Then $h > 0$ on $V_j \cap (V \cup V_K)$, where s_K is defined. By the section restriction property for V_j, there is a section s' of p_{V_j} extending

$$s_K|_{h^{-1}(1)} \supset A \cap V_j$$

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[24] As usual, a topological space is *paracompact* if every open cover admits a locally finite refinement. That is, some *subset* of the original covering opens can be made *smaller*, still covering the whole space (this is a *refinement*), and with the property that any compact subset meets only finitely many opens in the refined cover. It is a standard result that a second-countable locally compact Hausdorff space is paracompact. A moment's reflection shows that all CW-complexes are paracompact. It is more immediate that all compact Hausdorff spaces are paracompact.

[25] The terminology *vertical* is meant to suggest that it commutes with the map p.

[26] As usual, a *partition of unity* is a set of continuous functions taking values in $[0,1]$ and which sum to 1 at all points. The partition of unity $\{f_i\}$ is *subordinate* to a given cover $\{V_i : i \in I\}$ if they have the same indexing set and if the support of f_i is contained in V_i.

Then define a section s" of p over $V \cup V_K \cup V_j$ by

$$s"(b) = \begin{cases} s_K(b) & (\text{off } V_j) \\ s_K(b) & (\text{on } V_j \cap \{h(b) = 1\}) \\ s'(b) & (\text{on } V_j \cap \{h(b) < 1\}) \end{cases}$$

This gives $(K \cup \{j\}, s") \in S$, contradicting the maximality of S. That is, s_K is a global section extending $s|_A$, and B has the section extension property.

Now show that there is a *vertical homotopy* from 1_E to $s \circ p$. Already $p \circ s = 1_B$, so it is surely homotopic to the identity on 1_B, so this will prove that p is a homotopy equivalence. Let

$$E \times_B E = \{(x,y) \in E \times E : p(x) = p(y)\}$$

denote the usual fiber product. The argument so far shows that

$$q : E \times_B E \times [0,1] \longrightarrow E \times [0,1]$$

by

$$q(x,y,t) = (x,t)$$

has the section extension property. It is not claimed that E is paracompact. However, luckily, the covering of B used in the previous argument pulls back to a covering of $E \times [0,1]$ with a subordinate partition of unity. Let

$$A = E \times \{0,1\} \subset H = E \times \left([0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \right) \subset E \times [0,1]$$

This H is a halo around A, and a section of q is given by

$$s'(x,t) = \begin{cases} (x,x,t) & (\text{for } t < \frac{1}{4}) \\ (x, s \circ p(x), t) & (\text{for } t > \frac{3}{4}) \end{cases}$$

By the section extension property, the restriction of s' to A has a global extension. Composing this with projection onto the *second* copy of E, we obtain the desired (vertical) homotopy. ///

Theorem: *(Dold)* Let X, Y be contractible Hausdorff spaces on which a group Γ acts freely and properly discontinuously.[27] Suppose that both $\Gamma\backslash X$ and $\Gamma\backslash Y$ are *paracompact.* Then

$$\Gamma\backslash X \simeq \Gamma\backslash Y$$

that is, $\Gamma\backslash X$ and $\Gamma\backslash Y$ are *homotopy equivalent.*

Proof: For X Hausdorff and Γ acting freely and properly discontinuously, the map $X \longrightarrow \Gamma\backslash X$ is a local homeomorphism, is a covering map with Γ as covering group, and $\Gamma\backslash X$ is Hausdorff. For X locally contractible (for example), $\Gamma\backslash X$ is locally contractible with X as simply-connected covering space. Thus, X is the *universal* covering space of $\Gamma\backslash X$, with Γ as fundamental group.

Let $p : X \longrightarrow \Gamma\backslash X$ and $q : Y \longrightarrow \Gamma\backslash Y$ be the two covering maps. Let $Z = X \times Y$ with the diagonal action of Γ, namely

$$\gamma \cdot (x,y) = (\gamma x, \gamma y)$$

[27] In the requirement that G act freely and properly discontinuously on X, the terminology *properly discontinuously* is unfortunate terminology, but is standard. First, this assumes that the action of G is continuous, in the sense that the map $G \times X \longrightarrow X$ is continuous. The *proper discontinuous* requirement is that, for any $x \in X$, there is a small-enough neighborhood U of x such that $\gamma \in \Gamma$ with $C \cap \gamma C \neq \phi$ implies $\gamma = 1$.

This product is Hausdorff and contractible, and the action of Γ is free and properly discontinuous. If we show that the projection $p_X : Z \longrightarrow X$ induces a homotopy equivalence $p'_X \Gamma\backslash Z \longrightarrow \Gamma\backslash X$, then, symmetrically, $p_Y : Z \longrightarrow Y$ induces a homotopy equivalence $\Gamma\backslash Z \longrightarrow \Gamma\backslash Y$, giving the desired conclusion, by transitivity. Indeed, the induced map $p'_X \Gamma\backslash Z \longrightarrow \Gamma\backslash X$ meets the hypotheses of the previous theorem, with the X of the previous theorem taken to be the Y of the present, since Y is contractible, $\Gamma\backslash X$ is paracompact, and $\Gamma\backslash Z$ is locally a *product*. Thus, by the previous theorem, p'_X is a homotopy equivalence, and we're done. ///

5. Spaces $\Gamma\backslash X$

We want to consider spaces X homeomorphic to open balls, nicely acted upon by groups Γ, and show that quotients $\Gamma\backslash X$ are $K(\Gamma,1)$'s. We need only some mild hypotheses on Γ. Dold's theorem recalled in the previous section extends results on $K(G,1)$'s to quite general topological spaces, removing the need to make comparisons to CW-complexes. [28]

Even though Dold's theorem gives us great lee-way, in fact we are mostly interested in the following type of example. Let G be a semi-simple real Lie group, with no compact factors, such as $SL(n,\mathbf{R})$, $U(p,q)$ (with neither p nor q equal to 0), $Sp(n,\mathbf{R})$, or $O(p,q)$ (again with neither p nor q equal to 0). Let K be a maximal compact subgroup of G. There is an involution θ on G (the *Cartan involution*) such that K is the fixed-point set. [29] Let $X = G/K$, with the quotient topology. [30] Certainly G acts continuously on X. Let Γ be a torsion-free[31] discrete subgroup of G. [32]

Proposition: The subgroup Γ acts *freely and properly discontinuously* on X.

Proof:

EDIT: iou: but standard ///

Corollary: All homotopy invariants, including homology and cohomology of $\Gamma\backslash X$, depend functorially upon Γ alone.

Proof: Since X is contractible, and since Γ acts (continuously and) properly discontinuously, the natural map $p : X \longrightarrow \Gamma\backslash X$ presents X as a universal covering space of $\Gamma\backslash X$. Since X is second-countable, it and $\Gamma\backslash X$ are paracompact. We have also recalled the *functorial* construction of a simply-connected CW-complex $E\Gamma$ with quotient the classifying space $B\Gamma = \Gamma\backslash E\Gamma$. Then Dold's theorem says that the $\Gamma\backslash X$ is homotopy-equivalent to $B\Gamma$. In particular, the homotopy-equivalence type only depends upon Γ. ///

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[28] For example, we need not try to prove that $\Gamma\backslash X$ has a structure of CW-complex. And we need to worry about showing that spaces are *approximable* by CW-complexes, in the sense that their homotopy groups coincide. It is fortunate that we can avoid this, as this approximability is strictly weaker than homotopy equivalence, and, thus, proving that all homotopy *invariants* coincide would require yet more work.

[29] For all the classical groups mentioned explicitly, this and other properties are easy to prove directly. The general intrinsic (meaning not using classification) arguments require much more preparation regarding material not immediately helpful in other regards.

[30] For example, with $G = U(n,1)$ the maximal compact is $K = U(n) \times U(1)$, and $G/K \approx$ complex n-ball.

[31] Torsion-free-ness is sufficient for our purposes, although in other contexts stronger hypotheses are needed. This is especially true in trying to *compactify* non-compact quotients $\Gamma\backslash X$.

[32] At least for *arithmetic* subgroups such as $SL(n,\mathbf{Z})$ in $SL(n,\mathbf{R})$, various elementary conditions can guarantee the torsion-free property. For example, in $SL(n,\mathbf{Z})$, if Γ is contained in the subgroup of integer matrices congruent to 1_n modulo N for large-enough N, then no element of Γ other than 1_n has any eigenvalue a root of unity other than 1.

Remark: Still need to show that the geometric physical homology is *exactly* group homology. To this end, perhaps a discussion of *universal δ-functors* will be appropriate.

6. Appendix: contractibility of spheres

For dimension $n > 1$, the homotopy group $\pi_1(S^n)$ of the n-sphere is trivial. This is not trivial to prove, but, as we recall here, is easier than might be feared. By this time, the argument is standard, e.g., from [Hatcher 2002].

Theorem: For $n > 1$, $\pi_1(S^n) = \{1\}$.

Proof: If a map $f : S^1 \longrightarrow S^n$ misses any point x_0 of S^n, then a homeomorphism of $S^n - x_0$ to the contractible $\mathbf{R}^n$ shows that $f(S^1)$ is nullhomotopic. Thus, is suffices to alter a given f by a homotopy to arrange so that f is not a surjection. [33] Let $f : [0,1] \longrightarrow S^n$ be a continuous map with $f(0 = f(1) = x_0$. Let U be an open ball on the sphere centered at a given point $x \neq x_0$, small enough such that B does not contain x_0. The inverse image $f^{-1}(U)$ is open in the open interval $(0,1)$, so is expressible as a (possibly infinite) union of *disjoint* open intervals (a,b). The inverse image $f^{-1}(x)$ is closed in $[0,1]$, hence is compact, since $[0,1]$ is Hausdorff. Since $f^{-1}(x)$ cannot include the endpoints $0,1$, it is a compact subset of $(0,1)$, so is covered by finitely-many of the (a_i,b_i). Let (a,b) be one such interval meeting $f^{-1}(x)$. The little path $f|_{[a,b]}$ lies in the *closure* of B, the endpoints $f(a)$ and $f(b)$ are on the boundary of B, and $f(a,b)$ is in the interior. For $n > 1$, the closed ball $\overline{B}$ contains a path g from $f(a)$ to $f(b)$ that miss x. (The boundary of B itself, homeomorphic to S^{n-1}, is path-connected for $n > 1$.) The closure $\overline{B}$ is easily homeomorphic to a convex subset of $\mathbf{R}^n$, so $f|_{[a,b]}$ is homotopic to g. Performing this procedure for each of the finitely-many intervals (a_i,b_i) deforms f so that it misses x, then allowing a retraction to x_0. ///

7. Appendix: homotopy criterion for map lifting

For the reader's convenience, we recall the standard criteria for **lifting** of continuous maps via *coverings*.

As usual a continuous map $f : Z \longrightarrow Y$ is a **covering** if, for every $y \in Y$, there is a small-enough neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union of opens V_α such that on each V_α the map $f : V_\alpha \longrightarrow U$ is a homeomorphism. Let $I = [0,1]$ be the closed interval with the usual topology. Our topological spaces will be Hausdorff unless otherwise mentioned. For example, points are closed. The following proposition is the **homotopy lifting property**.

Proposition: Let $f\times : (X,x_0) \longrightarrow (Y,y_0)$ be a continuous map, with Y locally connected, and $p : (Z,z_0) \longrightarrow (Y,y_0)$ a covering. Let $F : X \times \{0\} \longrightarrow Z$ be continuous such that

$$p \circ F = f|_{X \times \{0\}}$$

Then there is a unique map $F : X \times I \longrightarrow Z$ extending the given F on $X \times \{0\}$.

Proof: By continuity, every point $x \times t$ in $X \times I$ has a neighborhood $V \times (a,b)$ such that the image $f(V \times (a,b))$ lies inside small-enough open U in Y such that $p^{-1}(U)$ meets the defining property of a covering, namely, $p^{-1}(U)$ is a disjoint union of opens on each of which p is a homeomorphism to U. For fixed $x \in X$, the set $\{x\} \times I$ is compact, so is covered by finitely-many of the products $U \times (a,b)$. Thus, we can choose a single neighborhood V of x and $0 < t_1 < \ldots < t_n < 1$ such that every $f(V \times (t_1, t_{i+1}))$ is contained inside a single small-enough open U_i in Y. Suppose that we have constructed a lifting on $V \times [0,t_i]$. Since $V \times [t_i, t_{i+1}]$ lies

[33] It would be foolish to presume that there is no continuous surjection of S^1 to S^n, although temporarily forgivable if one hasn't seen Peano's space filling curves.

inside a small-enough U_i, let W be an open subset of $p^{-1}(U_i)$ containing $F(x \times t_i)$ such that $p : W_i \longrightarrow U_i$ is a homeomorphism. Shrinking the neighborhood V of x if necessary, we can assume that $F(V \times t_i) \subset W_i$. Then define F on $V \times [t_i, t_{i+1}]$ as the composition

$$F|_{V\times[t_i,t_{i+1}]} = p^{-1}|_{U_i} \circ f|_{V\times[t_i,t_{i+1}]}$$

By a finite induction we obtain a lifting to $F : V \times I \longrightarrow Z$ for some open neighborhood V of a given point $x \in X$.

Next, prove uniqueness of these liftings to $\{x\} \times I$, that is, in the case that X is the single point $\{x\}$. Thus, in effect, let F and Φ be two continuous maps $I \longrightarrow Z$ such that

$$p \circ F = p \circ \Phi$$

As in the previous paragraph, choose a fine-enough partition $0 < t_1 < \ldots < t_n < 1$ such that each $[t_i, t_{i+1}]$ is contained in a small-enough open U_i in Y. We can assume that U_i is connected, by the local-connectedness of Y. Suppose F agrees with Φ on $[0, t_i]$. The connectedness of $[t_i, t_{i+1}]$ implies that of $F([t_i, t_{i+1}])$, so must lie in a single W_i of the connected components of $p^{-1}(U_i)$. Since $F(t_i) = \Phi(t_i)$, both $F([t_i, t_{i+1}])$ and $\Phi([t_i, t_{i+1}])$ lie in the same connected W_i. Since p is injective on W_i, the fact that $p \circ F = p \circ \Phi$ on $[t_i, t_{i+1}]$ implies that $F = \Phi$ on $[t_i, t_{i+1}]$. Thus, a finite induction gives the uniqueness of the extension.

Returning to the main argument, the little uniqueness result just established says that the functions constructed above on sets $V \times I$ are unique when restricted to sets $\{x\} \times I$. Thus, they agree on any overlaps of two of the sets $V \times I$, so piece together to give a well-defined (unique, continuous) lift on the whole $X \times I$. ///

The **path lifting property** for a covering space $p : (Z, z_0) \longrightarrow (Y, y_0)$ is obtained by taking $X = \{x\}$ in the previous:

Corollary: Each path in Y starting at y_0 lifts uniquely to a path in Z starting at z_0. ///

Theorem: Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a basepoint-preserving continuous map, with X path-connected and locally path-connected. Let $p : (Z, z_0) \longrightarrow (Y, y_0)$ be a *covering*. Then f factors through p if and only if

$$f_* \, \pi_1(X, x_0) \subset p_* \, \pi_1(Z, z_0)$$

Proof: If there exists a lifting F then $p \circ F = f$, and the induced map $p_* \circ F_* = f_*$ yields the indicated containment.

On the other hand, suppose that the indicated containment holds. Let γ be a path from x_0 to another point x in X. The path $f \circ \gamma$ from y_0 to $y = f(x)$ has a unique lifting $\delta_x : I \longrightarrow Z$ starting at z_0, by the path lifting property. We try to define a lifting

$$F : (X, x_0) \longrightarrow (Z, z_0)$$

of f by setting

$$F(x) = \delta_x(1)$$

To see that this is well-defined, let γ' be another path from x_0 to x and make a loop λ at x_0 by following γ from x_0 to x and following the inverse of γ' from x back to x_0. Then $f \circ \lambda$ gives an element $f_*([\lambda])$, which by hypothesis is inside $p_*(\pi_1(Z, z_0))$. That is, there is a homotopy h in Y (fixing y_0) of $f \circ \lambda$ to another loop μ at y_0 which is of the form $p \circ \nu$ for a loop ν in Z at z_0. The *covering homotopy property* (above) lifts the homotopy, yielding a loop ν' in Z at z_0 such that $p \circ \nu' = f \circ \lambda$.

By the uniqueness of the lifting, ν' is composed of δ_x followed by the *inverse* of δ'_x. That is,

$$\delta_x(1) = \delta'_x(1)$$

is well-defined.

For the continuity of the map $x \longrightarrow \delta_x(1)$, let U be a small-enough neighborhood of $f(x)$ so that $p^{-1}(U)$ contains an open V containing $F(x) = \delta_x(1)$ with $p : V \longrightarrow U$ a homeomorphism. Let W be a path-connected neighborhood of x such that $f(W) \subset U$. For other points $x' \in W$ we can take a fixed path γ from x_0 to x and then paths δ from x to x' within W. The paths $(f \circ \gamma) \cdot (f \circ \delta)$ lift to paths in Z, where the lift of $f \circ \delta$ is

$$p^{-1} \circ f \circ \delta$$

with $p^{-1} : U \longrightarrow V$ the inverse of the restriction of p to V. Then $F(W) \subset V$, which allows us to write $F|_W = p^{-1} \circ f$, which shows the continuity of F. ///

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