Archimedean Zeta Integrals

Paul Garrett, garrett@math.umn.edu
http://www.math.umn.edu/~garrett/

Background

Satake (mid-1960's) considered

$$G \to \tilde{G}$$

where G and $\tilde{G}$ are of hermitian type and the map is *of hermitian type* insofar as it respects this structure.

Then restriction of holomorphic automorphic forms from $\tilde{G}$ to G yields holomorphic things.

Shimura (mid-1970's) looked at examples

$$SL(2, \mathbf{Q}) \to Sp(n, \mathbf{Q})$$

$$SL(2, \mathbf{Q}) \to SL(2, F) \quad (F \text{ totally real})$$

wherein Fourier coefficients of restrictions are *finite sums* of Fourier coefficients on $\tilde{G}$, so a Fourier-coefficient-wise notion of *rationality* is preserved by restriction.

Shimura combined this with his *canonical models* results to give initiate the modern arithmetic of (holomorphic) automorphic forms. In particular, he generalized a classical principle:

For holomorphic Hecke eigenfunction f with totally real algebraic Fourier coefficients, and for g another holomorphic automorphic form with algebraic Fourier coefficients, *not necessarily a Hecke eigenfunction*,

$$\frac{\langle g, f \rangle}{\langle f, f \rangle} \in \overline{\mathbf{Q}}$$

and for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ the Galois equivariance

$$\left(\frac{\langle g, f \rangle}{\langle f, f \rangle} \right)^{\sigma} = \frac{\langle g^{\sigma}, f^{\sigma} \rangle}{\langle f^{\sigma}, f^{\sigma} \rangle}$$

with Galois acting on Fourier coefficients.

In the simplest application, $g = E \cdot h$ with E a holomorphic Eisenstein series and h a cuspform, and as in Rankin (who credits Ingham) for h a Hecke eigenfunction the integral *unwinds* giving a *special value* of an L-function

$$\langle E \cdot h, f \rangle = L(h \otimes f, s_o)$$

Combining the unwinding with the comparison of inner products gives

$$\frac{L(h \otimes f, s_o)}{\langle f, f \rangle} \in \overline{\mathbf{Q}}$$

and Galois equivariance.

To get *all* (or nearly all) predicted special values, Shimura took a *lower-weight* holomorphic Eisenstein series $E_{\rm low}$ and differentiated it to raise its weight before integrating.

$$\frac{L(h \otimes f,\ s_o - 2m)}{\langle f, f \rangle} = \frac{\langle D^m E_{\rm low} \cdot h, f \rangle}{\langle f, f \rangle} \in \overline{\mathbf{Q}}$$

Casting about for more examples: *Multiplicative* imbeddings

$$O(Q) \times Sp(V) \to Sp(Q \otimes V)$$

are not usually of hermitian type, but *additive* maps such as

$$Sp(V_1) \times Sp(V_2) \to Sp(V_1 \oplus V_2)$$

$$U(V_1) \times U(V_2) \to U(V_1 \oplus V_2)$$

are. In coordinates,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \to \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}$$

We want simple automorphic forms (or representations) to restrict and decompose. Not thetas, although they do interesting things under multiplicative imbeddings. Siegel-type (degenerate) Eisenstein series, now widely appreciated, were less popular circa 1980.

With holomorphy a complete decomposition (not just L^2) is possible (1980). Decomposing a holomorphic Siegel Eisenstein series along

$$Sp(m, \mathbf{Z}) \times Sp(n, \mathbf{Z}) \to Sp(m+n, \mathbf{Z})$$

$$\sum_{0 \le \ell \le \min(m,n)} \sum_{f \text{ cfm } Sp(\ell)} L(f, s_o) \frac{E_f^{(m)} \otimes E_f^{(n)}}{\langle f, f \rangle}$$

where $E_f^{(n)}$ is a Klingen-type Eisenstein series made from cuspform f on $Sp(\ell)$, and $L(f, s_o)$ is a special value of a standard L-function of f.

(Circa 1981, Böcherer explicated the L-function here, and at about the same time Rallis and Piatetski-Shapiro systematically treated the projection of the restriction of not-necessarily holomorphic degenerate Eisenstein series to cuspforms for classical groups, obtaining meromorphic continutations of standard L-functions.)

(The full decomposition also suggested that Klingen-type holomorphic Eisenstein series had an arithmetical nature, which was proven by Harris, 1981, 1982.)

To get as many *special values* as possible one must differentiate the Eisenstein series *transversally* before restricting.

Many have played this differentiate-restrict-and-integrate game, and/or restrict-differentiate-integrate.

The archimedean factors of these integrals are nasty to evaluate.

Unitary groups

After the preliminary unwinding and factoring over primes, one is left in situations like the following. Let

$$G = U(p,q) \qquad K = U(p) \times U(q)$$

$$\mathbf{p}_+ = \{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \in \mathbf{g}_{\mathbf{C}}\} \quad \mathbf{p}_- = \{\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \in \mathbf{g}_{\mathbf{C}}\}$$

We must evaluate integrals

$$Tf(g) = \int_G f(gh)\,\overline{\eta(h)}\,dh$$

η is left-annihilated by $\mathbf{p}_+$, right by $\mathbf{p}_-$, right K-type τ (descended from the Eisenstein series)

(cuspform) f right-annihilated by $\mathbf{p}_-$ generating holomorphic discrete series π_τ with extreme K-type τ

If $\eta \in L^1(G)$ then $f \to Tf$ is an endomorphism of π_τ not depending upon the model.

Unfortunately, integrability fails in the critical strip, necessitating a more complicated argument there... But let's suppose we have integrability.

The Harish-Chandra decomposition is

$$G \subset N_+ \cdot K_{\mathbf{C}} \cdot N_- \subset G_{\mathbf{C}}$$

with $N_\pm = \exp \mathbf{p}_\pm$. Thus,

$$f(g) = f_{u,v}(g) = f_{u,v}(n_+ \theta n_-) = c_{u,v}(\theta)$$

a matrix coefficient function.

For extreme K-type τ of sufficiently high extreme weight the universal $(\mathbf{g}, K)$-module generated by a vector v_τ of K-type τ and annihilated by $\mathbf{p}_-$ is irreducible. Thus, take

$$\eta_{\mu,\nu}(n_+ \theta n_-) = c_{\mu,\nu}(\theta)$$

Theorem:

$$Tf(1) = \int_G f_{u,v}(h)\, \overline{\eta_{\mu,\nu}(h)}\, dh$$

$$= \pi^{pq} \cdot \langle u, \mu \rangle \cdot \langle v, \nu \rangle \cdot (\text{rational number})$$

In particular, for example,

$$\tau(k_1 \times k_2) = (\det k_1)^m\, (\det k_2)^{-n} \quad (m \geq p,\ n \geq q)$$

the rational number is

$$\frac{\prod_{i=0}^{p+q-1} \Gamma(m+n-i)}{\prod_{i=0}^{p-1} \Gamma(m+n-p-i) \cdot \prod_{i=0}^{q-1} \Gamma(m+n-q-i)}$$

The real point here is not explicit evaluation, but illustration of a *qualitative* argument for the rationality of integrals.

We have a Cartan decomposition

$$G = C \cdot K \approx C \times K$$

where

$$C = \{g \in G = U(p,q) : g = g^* > 0\}$$

Parametrize C by

$$z \to g_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2} z^* & (1_q - z^*z)^{-1/2} \end{pmatrix}$$

where

$$D_{p,q} = \{z = p\text{-by-}q \text{ complex} : 1_p - zz^* > 0\}$$

$G = U(p,q)$ acts on $G/K \approx D_{p,q}$ with invariant measure

$$d^*z = \frac{dz}{\det(1_q - z^*z)^{p+q}} = \frac{dz}{\det(1_p - zz^*)^{p+q}}$$

To compute, use Cartan and Harish-Chandra, $h = h_z k$ and $h_z = n_z^+ \theta_z n_z^-$, where

$$h_z = \begin{pmatrix} (1_p - zz^*)^{-1/2} & z(1_q - z^*z)^{-1/2} \\ (1_q - z^*z)^{-1/2} z^* & (1_q - z^*z)^{-1/2} \end{pmatrix} =$$

$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} (1 - zz^*)^{1/2} & 0 \\ 0 & (1 - z^*z)^{-1/2} \end{bmatrix}}_{\theta_z} \begin{bmatrix} 1 & 0 \\ z^* & 1 \end{bmatrix}$$

The special form of $f_{u,v}$ gives

$$f_{u,v}(h_z k) = f_{u,v}(n_z^+ \theta_z n_z^- k) = f_{u,v}(\theta_z k \cdot k^{-1} n_z^- k)$$

and

$$f_{u,v}(\theta_z k) = \langle \tau(\theta_z k) u, v \rangle$$

and similarly for $\eta_{\mu,\nu}$. Suppressing τ,

$$Tf(1) = \int_C \int_K \langle \theta_z k \cdot u, v \rangle \, \overline{\langle \theta_z k \cdot \mu, \nu \rangle} \, dk \, d^* z$$

$$= \int_C \int_K \langle k \cdot u, \theta_z^* \cdot v \rangle \, \overline{\langle k \cdot \mu, \theta_z^* \cdot \nu \rangle} \, dk \, d^* z$$

Schur relations compute the integral over K

$$
\begin{aligned}
Tf(1) &= \frac{\langle u, \mu\rangle}{\dim \tau} \cdot \int_C \langle \theta_z^* \cdot \nu, \theta_z^* \cdot v\rangle \, d^* z \\
&= \frac{\langle u, \mu\rangle}{\dim \tau} \cdot \langle \nu, \ \int_C \tau(\theta_z^2)\, d^* z \cdot \ v\rangle
\end{aligned}
$$

since $\tau(g^*) = \tau(g)^*$ for g in $K_{\mathbf{C}}$, and $\theta_z^* = \theta_z$. We compute the endomorphism

$$
S = S(\tau) = \int_C \tau(\theta_z^2)\, d^* z
$$

where

$$
\theta_z^2 = \begin{pmatrix} 1_p - zz^* & 0 \\ 0 & (1_q - z^* z)^{-1} \end{pmatrix}
$$

$\tau \approx \tau_1 \otimes \tau_2$ with irreducibles τ_1 of $U(p)$ and τ_2 of $U(q)$, so

$$
S = \int_{D_{p,q}} \tau_1(1_p - zz^*) \otimes \tau_2^{-1}(1_q - z^* z)\, d^* z
$$

Mapping $z \to \alpha z \beta^*$ with $\alpha \in U(p)$, $\beta \in U(q)$ in the integral shows that S commutes with $\tau(k)$, so by Schur's lemma S is *scalar*.

Let $z = \alpha r \beta$ with $\alpha \in U(p)$, $\beta \in U(q)$, and

$$r = \text{ } p\text{-by-}q \text{ } = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_q \\ 0 & \dots & 0 \end{pmatrix}$$

with $-1 < r_i < 1$. Let $\Delta(r) = \prod_{i<j} \left(r_i^2 - r_j^2\right)^2$
Up to a constant C (determined subsequently)

$$\int_{D_{p,q}} h(z) \, \frac{dz}{\det(1_q - z^* z)^{p+q}}$$

$$= C \cdot \int \int_{(-1,1)^q} h(\alpha r \beta) \, d\alpha \, d\beta \, \frac{\Delta(r) \, dr}{\det(1_q - r^* r)^{p+q}}$$

Thus, S is

$$C \cdot \int_{U(p) \times U(q)} (\alpha \otimes \beta) \cdot I \cdot (\alpha \otimes \beta)^{-1} d\alpha \, d\beta$$

where the inner integral I is

$$I = \int_{(-1,1)^q} (1 - r r^*) \otimes (1 - r^* r)^{-1} \, \frac{\Delta(r) \, dr}{\det(1 - r^* r)^{p+q}}$$

The inner integral I in S acts on weight spaces by scalars. The identity

$$(t^2 - u^2) = (t^2 - 1) - (u^2 - 1)$$

shows that each such scalar is a $\mathbf{Q}$-linear combination of products of integrals

$$\int_{-1}^{1} (1 - t^2)^n \frac{dt}{(1 - t^2)^{p+q}}$$

$$= 2^{2n+1-p-q} \frac{\Gamma(n-p-q+1)\,\Gamma(n-p-q+1)}{\Gamma(2n-2p-2q+2)}$$

$$= \text{ rational}$$

so the inner integral I acts by rational scalars on all weight spaces. In particular, I so is a rational endomorphism of τ.

(Better give τ a rational structure...)

The outer integration is the projection

$$\mathrm{End}_{\mathbf{C}}(\tau) \to \mathrm{End}_K(\tau)$$

where $\mathrm{End}_{\mathbf{C}}(\tau)$ has the K-structure

$$k \cdot \varphi = \tau(k) \circ \varphi \circ \tau(k)^{-1}$$

$\mathrm{End}_{\mathbf{C}}(\tau)$ has a rational structure compatible with

$$\mathbf{g}_{\mathbf{Q}} = \mathbf{gl}(p, \mathbf{Q}) \otimes \mathbf{gl}(q, \mathbf{Q})$$

on the complexified Lie algebra

$$\mathbf{g}_{\mathbf{C}} = \mathbf{gl}(p, \mathbf{C}) \otimes \mathbf{gl}(q, \mathbf{C})$$

of $K = U(p) \times U(q)$.

Poincaré-Birkhoff-Witt, the Harish-Chandra homomorphism, and Verma modules still work over $\mathbf{Q}$.

Highest weights $\lambda - \rho$ for finite-dimensional irreducibles are *integral* (and dominant), so are rational on a rational Cartan subalgebra. Give a finite-dimensional irreducible complex representation τ a rational structure

$$\tau = (M_\lambda / N_\lambda) \otimes_{\mathbf{Q}} \mathbf{C}$$

with *rational* Verma module M_λ and (unique) maximal proper submodule N_λ.

$Z(\mathbf{g}_{\mathbf{Q}})$ distinguishes finite-dimensional irreducibles: given finite-dimensional irreducibles V and V' with highest weights $\lambda - \rho = \lambda' - \rho$, there is $z \in Z(\mathbf{g}_{\mathbf{Q}})$ such that $z(\lambda) \neq z(\lambda')$.

Let Λ be the finite collection of λ's indexing irreducibles in $\mathrm{End}_{\mathbf{C}}(\tau) = \mathrm{End}_{\mathbf{Q}}(\tau_{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{C}$. Then

$$P = \prod_{\lambda \in \Lambda} z_\lambda \ \in \ Z(\mathbf{g}_{\mathbf{Q}})$$

projects endomorphisms to the K-invariants. Thus, projection to K-endomorphisms preserves rationality.

To determine C compute

$$S = S_\tau = \int_{D_{p,q}} \det(1_p - zz^*)^m \det(1_q - z^*z)^{-n} d^*z$$

For $0 < \ell \in \mathbf{Z}$, let

$$C_\ell = \{\ell\text{-by-}\ell \text{ complex } Y > 0\}$$

For real $s > \ell - 1$ define

$$\Gamma_\ell(s) = \int_{C_\ell} e^{-\operatorname{tr} x} (\det x)^s \frac{dx}{(\det x)^\ell}$$

$$= \pi^{\ell(\ell-1)/2} \prod_{i=1}^{\ell} \Gamma(s - i + 1)$$

Imitating classical computations,

$$\Gamma_p(m+n-p)\Gamma_q(m+n-q) \cdot S =$$

$$\int_{C_{p+q}} e^{-\operatorname{tr} Z} (\det Z)^{m+n} \frac{dZ}{(\det Z)^{p+q}} = \Gamma_{p+q}(m+n)$$

Thus, for this τ

$$S = \frac{\Gamma_{p+q}(m+n)}{\Gamma_p(m+n-p)\,\Gamma_q(m+n-q)} =$$

$$\frac{\prod_{i=0}^{p+q-1}\Gamma(m+n-i)}{\prod_{i=0}^{p-1}\Gamma(m+n-p-i)\cdot\prod_{i=0}^{q-1}\Gamma(m+n-q-i)}$$

$$\times$$

$$\frac{\pi^{(p+q)(p+q-1)/2}}{\pi^{p(p-1)/2}\cdot\pi^{q(q-1)/2}}$$

The net exponent of π is

$$(p+q)(p+q-1)/2 - p(p-1)/2 - q(q-1)/2 = pq$$

as anticipated. Thus,

$$C = \pi^{pq}\cdot(\text{rational})$$

and for *arbitrary* τ

$$S = \pi^{pq}\cdot(\text{rational scalar endomorphism of } \tau)$$
