Corrected version of "Banach-space criterion" is in http://www.math.umn.edu/~garrett/m/v/Bernstein_continuation_principle.pdf

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Meromorphic continuation of Eisenstein series for $SL(2)$

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Draft

As Bernstein has commented, the germ of this idea for meromorphic continuation of certain Eisenstein series is the so-called 'third proof' of A. Selberg (described in SLN 1001, appendix F). Bernstein apparently intended application not only to higher-rank cases but also to other sorts of problems entirely. However, there are details which needed working out to give complete proofs for Eisenstein series on higher-rank groups. See also A. Borel's book on automorphic forms on $SL(2,\mathbf{R})$ for a modification of Bernstein's argument (in that special case) which avoids *certain* complications by reverting to a more traditional argument.

The idea of the continuation principle is as follows. We use terminology whose *intent* should be clear, but whose *definition* is only given below. Let X_s be a system of linear equations depending holomorphically upon a parameter s in a connected complex manifold. Suppose that for all s in some non-empty open set this system has a *unique* solution v_s. Then the **continuation principle** would assert this solution v_s extends to a meromorphic function $s \to v_s$ on the whole manifold. Moreover, almost everywhere (in fact, outside a proper analytic subset) v_s is the unique solution of X_s. That is, the principle would be that **unique holomorphic characterization assures meromorphic continuation**.

The present notes represent an attempt to pay attention to all details, and to formulate and prove things in as smooth and extensible manner as possible. In particular, this requires inclusion of some facts about vector-valued integrals and vector-valued holomorphic functions which deserve to be considered 'standard', but probably are not.

Thanks to D. Hejhal and P. Sarnak for notes of a lecture given by J. Bernstein possibly in 1984, maybe at Stanford.

Weak-to-strong principles

Definition: Let V be a topological vectorspace. A V-valued function $s \to f(s)$ is **holomorphic** at a point s_o if

$$\lim_{s\to s_o} \frac{f(s)-f(s_o)}{s-s_o} \quad \text{exists}$$

It is simply **holomorphic** if it is holomorphic pointwise everywhere.

Definition: Let V be a topological vectorspace. A V-valued function $s \to f(s)$ is **weakly holomorphic** if for every continuous linear functional λ on V the $\mathbf{C}$-valued function $s \to \lambda(f(s))$ is holomorphic.

An immediate goal is to determine useful contexts in which *weak holomorphy* implies *holomorphy*. To emphasize the distinction, we may use the phrase *strong holomorphy* rather than simply *holomorphy*. The issue then is that of **weak-to-strong** principles.

Definition: A family of operators $T_s : V \to W$ from one topological vectorspace to another is **weakly holomorphic** in a parameter s (in a connected complex manifold, for example a connected open subset of $\mathbf{C}$) if for every vector $v \in V$ and for every continuous functional $\mu \in W^*$ the $\mathbf{C}$-valued function $\mu(T_s v)$ is holomorphic in s.

Remark: A virtue of this definition is that it seems not to explicitly mention a topology on the space of operators from V to W. In fact, of course, there *is* a topology implicity defined, namely the **weak operator topology**.

Before we address weak-to-strong problems, we can exploit a different mechanism (Hartogs' theorem in several complex variables) to note an important fact:

Proposition: Let $S_s : A \to B$ and $T_s : B \to C$ be two weakly holomorphic families of continuous linear operators on topological vectorspaces A, B, C. Then the composition $T_s \circ S_s : A \to C$ is weakly holomorphic. Similarly, for a weakly holomorphic A-valued function $s \to f(s)$ and for a weakly holomorphic continuous linear map $S_s : A \to B$, the composite $S_s \circ f$ is a weakly holomorphic B-valued function.

Proof: This is an immediate corollary of Hartogs' theorem that separate analyticity of a function of several complex variables implies joint analyticity (without any other hypotheses). Specifically, consider the family of operators

$$T_t \circ S_s$$

By the definition of weak holomorphy, for $x \in A$ and a linear functional $\mu \in C^*$ the $\mathbf{C}$-valued function

$$(s,t) \to \mu(T_t(S_s(x))$$

is separately analytic. By Hartogs' theorem, it is jointly analytic. It follows that the diagonal function

$$s \to (s,s) \to \mu(T_s(S_s(x))$$

is analytic. The same prove suffices for the second assertion. ♣

It should be striking that no hypotheses on the topological vectorspaces are necessary for the latter composition closure property of weakly holomorphic maps. By contrast, the rest of this section will be devoted to inference of *genuine holomorphy* from *weak holomorphy.*

Definition: A **Gelfand-Pettis** or **weak** integral of a function $s \to f(s)$ on a measure space (X, μ) with values in a topological vectorspace V is an element $I \in V$ so that for all $\lambda \in V^*$

$$\lambda(I) = \int_X f(s)\, d\mu(s)$$

Definition: A topological vectorspace is **quasi-complete** if every *bounded* (in the topological vectorspace sense, not necessarily the metric sense) Cauchy *net* is convergent.

Proposition: Continuous compactly-supported functions $f : X \to V$ with values in *quasi-complete* (locally convex) topological vectorspaces V have Gelfand-Pettis integrals with respect to finite positive regular Borel measures μ on compact spaces X, and these integrals are *unique.* In particular, for a measure μ with total measure $\mu(X) = 1$, the integral $\int_X f(x)\, d\mu(s)$ lies in the closure of the convex hull of the image $f(X)$ of the measure space X.

Proof: See Bourbaki's *Integration.* Also, in Rudin's *Functional Analysis* it is proven that the previous proposition holds for V in which the convex hull of a compact set has compact closure, and in Bourbaki's *Topological Vectorspaces* it is proven that quasi-complete spaces have the latter property. In any case, the uniqueness follows from the local convexity, by invoking the Hahn-Banach theorem. ♣

The following property of Gelfand-Pettis integrals is broadly useful in applications, such as justifying differentiation under integrals.

Proposition: Let $T : V \to W$ be a continuous linear map, and let $f : X \to V$ be a continuous compactly supported V-valued function on a topological measure space X with finite positive Borel measure μ. Suppose that V is locally convex and quasi-complete, so that (from above) a Gelfand-Pettis integral of f exists and is unique. Suppose that W is locally convex. Then

$$T\left(\int_X f(x)\,d\mu(x)\right) = \int_X Tf(x)\,d\mu(x)$$

In particular, $T\left(\int_X f(x)\,d\mu(x)\right)$ is a Gelfand-Pettis integral of $T \circ f$.

Proof: First, the integral exists in V, from above. Second, for any continuous linear functional λ on W, $\lambda \circ T$ is a continuous linear functional on V. Thus, by the defining property of the Gelfand-Pettis integral, for every such λ

$$(\lambda \circ T)\left(\int_X f(x)\,d\mu(x)\right) = \int_X (\lambda T f)(x)\,d\mu(x)$$

This exactly asserts that $T\left(\int_X f(x)\,d\mu(x)\right)$ is a Gelfand-Pettis integral of the W-valued function $T \circ f$. Since the two vectors $T\left(\int_X f(x)\,d\mu(x)\right)$ and $\int_X Tf(x)\,d\mu(x)$ give identical values under all $\lambda \in W^*$, and since W is locally convex, these two vectors are equal, as claimed. ♣

Corollary: Let V be quasi-complete (and locally convex). Then weakly holomorphic V-valued functions are (strongly) holomorphic.

Proof: The Cauchy integral formulas involve continuous integrals on compacta, so these integrals exist as Gelfand-Pettis integrals. Thus, we can obtain V-valued convergent power series expansions for weakly holomorphic V-valued functions, from which (strong) holomorphy follows by an obvious extension of Abel's theorem that analytic functions are holomorphic. See also Rudin's *Functional Analysis* in which the hypothesis that Gelfand-Pettis integrals exist is observed to be sufficient to reach this conclusion. ♣

Remark: The fact that for Banach-space valued functions weak holomorphy implies holomorphy is better-known, but inadequate for applications, and not much simpler to prove than the more general fact. Of course, in practice many spaces can be presented as limits or colimits of Banach spaces, and sometimes weak-to-strong questions can be reduced to the Banach space case. However, the general result using quasi-completeness is simpler and easier.

Corollary: Give the space $\mathrm{Hom}^o(X,Y)$ of continuous mappings $T : X \to Y$ from an LF space X (strict colimit of Fréchet spaces, e.g., a Fréchet space) to a quasi-complete space Y the natural *weak operator* topology as follows. For $x \in X$ and $\mu \in Y^*$, define a seminorm $p_{x,\mu}$ on $\mathrm{Hom}^o(X,Y)$ by

$$p_{x,\mu}(T) = |\mu(T(x))|$$

Then with this topology $\mathrm{Hom}^o(X,Y)$ is quasi-complete.

Proof: The collection of finite linear combinations of the functionals

$$T \to \mu(Tx)$$

is exactly the dual space of $\mathrm{Hom}^o(X,Y)$ when it is given the weak operator topology, so we can invoke the previous result. ♣

Corollary: Let $s \to T_s$ be a weakly holomorphic $\mathrm{Hom}^o(X,Y)$-valued function, meaning that for every $x \in X$ and $\mu \in Y^*$ the $\mathbf{C}$-valued function

$$s \to \mu(T_s(x))$$

is holomorphic. Then $s \to T_s$ is holomorphic when $\mathrm{Hom}^o(X,Y)$ is given the weak operator topology.

Proof: Collect the results above. ♣

In fact, a more general result asserting quasi-completeness of spaces of continuous linear mappings holds, of which the above corollary is the special case in which the target space Y is given its weak topology.

Proposition: Give the space $\mathrm{Hom}^o(X, Y)$ of continuous mappings $T : X \to Y$ from an LF space X (strict colimit of Fréchet spaces, e.g., a Fréchet space) to a quasi-complete space Y the natural *strong operator* topology as follows. For $x \in X$ and a convex balanced open subset U of Y^*, define a seminorm $p_{x,\mu}$ on $\mathrm{Hom}^o(X, Y)$ by

$$p_{x,U}(T) = \inf \{t > 0 : Tx \in tU\}$$

Then with this topology $\mathrm{Hom}^o(X, Y)$ is quasi-complete. ♣

Meromorphic continuation principle

Let V be a topological vector space. A **system of linear equations** X_o **in** V is a collection

$$X_o = \{(V_i, T_i, v_i) : i \in I\}$$

where I is a (not necessarily countable) set of indices, each V_i is a topological vector space,

$$T_i : V \to V_i$$

is a continuous linear map for each index i, and $v_i \in V_i$ are the 'targets'. A **solution** of the system X_o is a vector $v \in V$ such that $T_i(v) = v_i$ for all indices i. The set of solutions is denoted by $\mathrm{Sol\,X_o}$.

Now let the systems of linear equations $X_s = \{V_i, T_{i,s}, v_{i,s}\}$ depend on a parameter s varying in a connected complex manifold, *the parameter space*. Say that the **parametrized linear system** $X = \{X_s : s \in S\}$ is **holomorphic** in s if $T_{i,s}$ and $v_{i,s}$ are weakly holomorphic in s. (Note that $\{V_i\}$ does not depend upon s.)

Definition: Let $X = \{X_s\}$ be a parametrized system of linear equations in a space V, holomorphic in s. Suppose that there is a finite-dimensional space F and a weakly holomorphic family of continuous linear maps $f_s : F \to V$ such that, for each s, $\mathrm{Im}\, f_s \supset \mathrm{Sol\,X_s}$. Then say that $\{f_s\}$ is a **finite holomorphic envelope** for the system X, and that X is of **finite type**.

Definition: Let $U_\alpha, \alpha \in A$ be an open cover of the parameter space. Suppose that for each $\alpha \in A$ we have a finite envelope $\{f_s^{(\alpha)} : s \in U_\alpha\}$ for the system

$$X^{(\alpha)} = \{X_s : s \in U_\alpha\}$$

Then the collection

$$\{f_s^{(\alpha)} : s \in U_\alpha, \alpha \in A\}$$

is said to be a **locally finite holomorphic envelope** of X.

Remark: If there is a meromorphic continuation v_s of a solution, then by taking $F = \mathbf{C}$ and

$$f_s : \mathbf{C} \to V$$

to be

$$f_s(z) = z \cdot v_s$$

we trivially obtain a finite holomorphic envelope for parameter values s away from the poles of v_s. That is, if there is a meromorphic continuation, then for trivial reasons there is a finite holomorphic envelope, and the system is of finite type.

Theorem: Continuation Principle: Let $X = \{X_s : s\}$ be a *locally finite* system of linear equations

$$T_{i,s} : V \to V_i$$

for s varying in a connected complex manifold. Suppose that each V_i is (locally convex and) **quasi-complete.** Then the **continuation principle** holds. That is, if for s in some non-empty open subset there is a unique solution v_s, then this solution depends meromorphically upon s, has a meromorphic continuation to s in the whole manifold, and for fixed s off a locally finite set of analytic hypersurfaces inside the complex manifold, the solution v_s is the *unique* solution to the system X_s.

Proof: It is sufficient to check the continuation principle locally, so let $f_s : F \to V$ be a family of morphisms so that $\mathrm{Im}\, f_s \supset \mathrm{Sol\, X_s}$, with F finite-dimensional. We can reformulate the statement in terms of the finite-dimensional space F. Namely, put

$$K_s^+ = \{v \in F : f_s(v) \in \mathrm{Sol\, X_s}\} = \{\text{ inverse images in } F \text{ of solutions }\}$$

(The set K_s^+ is an affine subspace of F.) By elementary finite-dimensional linear algebra, X_s has a unique solution if and only if

$$\dim K_s^+ = \dim \ker f_s$$

The weak holomorphy of $T_{i,s}$ and f_s yield the weak holomorphy of the composite $T_{i,s} \circ f_s$ from the finite-dimensional space F to V_i, by the corollary of Hartogs' theorem above. The finite-dimensional space F is certainly LF, and V_i is quasi-complete, so by invocation of results above on weak holomorphy the space $\mathrm{Hom}^o(F, V_i)$ is quasi-complete, and a weakly holomorphic family in $\mathrm{Hom}^o(F, V_i)$ is in fact holomorphic.

Take $F = \mathbf{C}^n$. Using linear functionals on V and V_i which separate points we can describe $\ker f_s$ and K_s^+ by systems of linear equations of the forms

$$\ker f_s = \{(x_1, \ldots, x_n) \in F : \sum_j a_{\alpha\, j}\, x_j = 0, \ \ \alpha \in A\}$$

$$K_s^+ = \{\text{ inverse images of solutions }\} = \{(x_1, \ldots, x_n) \in F : \sum_j b_{\beta\, j}\, x_j = c_\beta, \ \ \beta \in B\}$$

where $a_{\alpha\, j}$, $b_{\beta\, j}$, c_β all depend implicitly upon s, and are holomorphic $\mathbf{C}$-valued functions of s. (The index sets A, B may be of arbitrary cardinality.) Arrange these coefficients into matrices M_s, N_s, Q_s holomorphically parametrized by s, with entries

$$M_s(\alpha, j) = a_{\alpha\, j} \qquad N_s(\beta, j) = b_{\beta\, j} \qquad Q_s(\beta, j) = \begin{cases} b_{\beta\, j} & \text{for } 1 \le j \le n \\ c_\beta & \text{for } j = n \end{cases}$$

of sizes $\mathrm{card}\,(A)$-by-n, $\mathrm{card}\,(B)$-by-n, $\mathrm{card}\,(B)$-by-$(n+1)$. We have

$$\dim \ker f_s = n - \mathrm{rank}\, M_s$$

Certainly for all s

$$\mathrm{rank}\, N_s \le \mathrm{rank}\, Q_s$$

and if the inequality is *strict* then there is *no solution* to the system X_s. By finite-dimensional linear algebra, assuming that $\mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$,

$$\dim K_s^+ = n - \mathrm{rank}\, B_s$$

Therefore, the condition that $\dim K_s^+ = \dim \ker f_s$ can be rewritten as

$$\mathrm{rank}\, M_s = \mathrm{rank}\, N_s = \mathrm{rank}\, Q_s$$

Let S_o be the dense subset (actually, S_o is the complement of an analytic subset) of the parameter space where $\mathrm{rank}\, M_s$, $\mathrm{rank}\, N_s$, and $\mathrm{rank}\, Q_s$ all take their maximum values. Since by hypothesis $S_o \cap \Omega$ is not empty, and since the ranks are equal for $s \in \Omega$, all those maximal ranks are equal to the same number r. Then for all $s \in S_o$ the rank condition holds and X_s has a solution, and the solution is unique.

In order to prove the continuation principle we must show that $X = \{X_s\}$ has a meromorphic solution v_s. Making use of the finite envelope of the system X, to find a meromorphic solution of X it is enough to find a meromorphic solution of the parametrized system

$$Y = \{Y_s\}$$

where

$$Y_s = \{\sum b_{\beta\, i} x_i = c_\beta : \text{ for all } \beta\}$$

with implicit dependence upon s. Let r be the maximum rank, as above. Choose $s_o \in S_o$ and choose an r-by-r minor

$$D_{s_o} = \{b_{\beta,j} : \beta \in \{\beta_1, \ldots, \beta_r\}, j \in \{j_1, \ldots, j_r\}\}$$

of full rank, inside the matrix N_{s_o}, with constraints on the indices as indicated. Let $S_1 \subset S_o$ be the set of points s where D_s has full rank, that is, where $\det D_s \neq 0$. Consider the system of equations

$$Z = \{ \sum_{j \in \{j_1, \ldots, j_r\}} b_{\beta\, j}\, x_j = c_\beta : \beta \in \{\beta_1, \ldots, \beta_r\}\} \quad \text{(with } s \text{ implicit)}$$

By Cramer's Rule, for $s \in S_1$ this system has a unique solution $(x_{1,s}, \ldots, x_{r,s})$. Further, since the coefficients are holomorphic in s, the expression for the solution obtained via Cramer's rule show that the solution is meromorphic in s. Extending this solution by $x_j = 0$ for j not among $j_1, \ldots, j_r$, we see that it satisfies the r equations corresponding to rows $\beta \in \{\beta_1, \ldots, \beta_r\}$ of the system Y_s. Then for $s \in S_1$ the equality $\text{rank} N_s = \text{rank} Q_s = r$ implies that after satisfying the first r equations of Y_s it will automatically satisfy the rest of the equations in the system Y_s.

Thus, the system has a *weakly* holomorphic solution. Earlier observations on weak-to-strong principles assure that this solution is holomorphic. This proves the continuation principle. ♣

Criteria for finiteness

Proposition: (**Dominance**) (Called *inference* by Bernstein.) Let $X' = \{X'_s\}$ be another holomorphically parametrized system of equations in a linear space V', with the same parameter space as a system $X = \{X_s\}$ on a space V. We say that X' **dominates** X if there exists a family of morphisms $h_s : V' \to V$, weakly holomorphic in s, so that for each s

$$\mathrm{Sol\,X_s} \subset \mathrm{h_s(Sol\,X'_s)}$$

If X'_s is locally finite then X_s is locally finite.

Proof: The fact that compositions of weakly holomorphic mappings are weakly holomorphic assures that X'_s really meets the definition of 'system'. Granting this, the conclusion is clear. ♣

Remark: The following criterion has a delicate feature: it requires holomorphy in the *uniform-norm* topology on operators, not in the weak operator topology.

Proposition: (**Banach-space criterion**) Suppose V is a Banach space and X is given by one parametrized homogeneous equation $T_s(v) = 0$, with $T_s : V \to W$, where W is also a Banach space, and where $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology on $\mathrm{Hom}^o(V, W)$. Suppose that for some fixed s_o there exists an operator $A : W \to V$ so that the composite

$$A \circ \lambda_{s_o}$$

has *finite-dimensional kernel* V_o and *closed image* V_1. Then X_s is of finite type in some neighborhood of s_o.

Proof: Let V_1 be the image of $A \circ T_{s_o}$, and let V_o be the kernel of $A \circ T_{s_o}$. By the Hahn-Banach theorem there exist continuous linear maps $\mathrm{pr}_{V_o} : V \to V_o$ and $\mathrm{pr}_{V_1} : V \to V_1$ which are 'projections' in the weak sense that for $i = 1, 2$

$$\mathrm{pr}_{V_i}\big|_{V_i} = 1_{V_i}$$

Consider a new system X'_s in V, given by the equation

$$T'_s(v) = 0$$

where

$$T'_s = \mathrm{pr}_{V_1} \circ A \circ T_s : V \to V_1$$

Since every solution of X is a solution of X', X' dominates X. Consider the family of maps

$$\varphi_s = \mathrm{pr}_{V_o} \oplus T'_s : V \to V_o \oplus V_1$$

where $V_o \oplus V_1$ is given the natural Banach-space topology. Since the family $s \to A \circ T_s$ is continuous for the uniform-norm topology on $\mathrm{Hom}^o(V, V)$, the same is true of the family $s \to \varphi_s$. By construction, φ_{s_o} is a bijection, and by the open-mapping theorem φ_{s_o} is an isomorphism. The (continuous) inverse $\varphi_{s_o}^{-1}$ has an operator norm $0 < \delta^{-1} < +\infty$. For s sufficiently near s_o so that

$$|\varphi_{s_o} - \varphi_s| < \frac{\delta}{2}$$

we find that

$$|\varphi_s(x)| \geq |\varphi_{s_o}(x)| - |\varphi_s(x) - \varphi_{s_o}(x)| \geq \delta \cdot |x| - \frac{\delta}{2} \cdot |x| \geq \frac{\delta}{2} \cdot |x|$$

This shows that φ_s is an isomorphism for s in a neighborhood of s_o.

Next, we show that the map $s \to \varphi_s^{-1}$ is holomorphic on a neighborhood of s_o. To do this, it suffices to observe that the uniform norm topology on $\mathrm{Hom}^o(V, V_o \oplus V_1)$ when restricted to the subset of invertible elements is the same as the uniform norm topology on $\mathrm{Hom}^o(V_o \oplus V_1, V)$ restricted to invertible elements. And, indeed, this is so, because on a sufficiently small neighborhood of an invertible map T_o the inversion map $T \to T^{-1}$ is continuous, by an elementary Banach-space computation.

Now we can easily see that there is a finite envelope for the parametrized system X':

$$\mathrm{Sol}\,\mathrm{X}'_\mathrm{s} = \varphi_\mathrm{s}^{-1}(\mathrm{V_o} \oplus \{0\})$$

and V_o is finite-dimensional by hypothesis. ♣

Corollary: (**Compact operator criterion**) Suppose that V is a Banach space and that X is given by one parametrized homogeneous equation $T_s(v) = 0$, with $T_s : V \to W$, where W is also a Banach space, and where $s \to T_s$ is holomorphic for the uniform-norm Banach-space topology on $\mathrm{Hom}^o(V, W)$. Suppose that for some fixed s_o the operator T_{s_o} has a left inverse modulo compact operators, that is, that there exists an operator $A : W \to V$ so that

$$A \circ \lambda_{s_o} = 1_V + (\text{compact operator})$$

Then X_s is of finite type in some neighborhood of s_o.

Proof: By compact operator theory theory V_o is finite-dimensional. Another compact-operator argument shows that V_1 is necessarily a closed subspace of V. (By Fredholm theory, in fact V_1 is of finite codimension equal to the dimension of V_o, but we don't need this). This presents the hypotheses of the previous criterion. ♣

Example: Eisenstein series for $SL(2, \mathbf{Z})$

We illustrate the use of the continuation principle via the simplest Eisenstein series, namely, by proving meromorphic continuation of the spherical Eisenstein series for the discrete subgroup $\Gamma = SL(2, \mathbf{Z})$ of $G = SL(2, \mathbf{R})$. Let $K = SO(2) \subset G$ be the standard maximal compact subgroup. We restrict our attention to right K-invariant functions on G. Define

$$N = \{n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R}\} \quad A^+ = \{a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} : y > 0\} \quad A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}^\times\}$$

and let

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} = AN \subset G$$

Put $\Gamma = SL(2, \mathbf{Z})$. As usual, for complex s, define a $\mathbf{C}$-valued function $x \to a_x^s$ on G by

$$a^s_{n a_y k} = y^s$$

with $y > 0$, $n \in N$, $k \in K$. We may simply write a^s for the function $x \to a_x^s$.

Siegel sets S_t in G are subsets of the form

$$S_t = \{n_x a_y K : x \in [0, 1], y \geq t\}$$

for fixed $t > 0$. Classical 'reduction theory' asserts that for any $0 < t \leq \sqrt{3}/2$,

$$\Gamma \cdot S_t = G$$

For a smooth function φ of moderate growth on $(\Gamma \cap P)\backslash G$, define an **Eisenstein series** attached to it by

$$E(\varphi)(g) = \sum_{\Gamma \cap P \backslash \Gamma} \varphi(\gamma\, g)$$

On the other hand, for a function f on $(\Gamma \cap N)\backslash G$ define its **constant term** $c_P f$ along P by

$$c_P f(g) = \int_{\Gamma\cap N\backslash N} f(ng)\, dn$$

Remark: The maps E and c commute with the right regular representation of G on functions on G, since both formation of Eisenstein series and formation of constant terms involve action on the *left*.

For $\mathrm{Re}\,(s) > 1$, define the simplest Eisenstein series by $E_s(g) = E(a_g^s)$.

The series for E_s converges absolutely and uniformly on compacta, in $g \in G$ and $s \in \mathbf{C}$ for $\mathrm{Re}\, s \geq \sigma_o > 1$.

A little work (recapitulating Godement's proof of convergence of Eisenstein series) shows that (for $\mathrm{Re}\,(s) > 1$) E_s is a function of *moderate growth*, i.e., for large N, $a^{-N} E_s$ is bounded on every Siegel set S_t.

Proposition: The constant term $c_P E_s$ of E_s is of the form

$$c_P E_s = a^s + b a^{1-s}$$

for a constant b depending upon s.

Proof: (*Sketch*) To understand this properly it is better to look at the Eisenstein series on the adele group rather than the real Lie group. Then each of the two Bruhat cells contributes a summand to this constant term, with the small cell returning simply a^s, while the big cell presents an integral

$$\int_{N_{\mathbf{A}}} a^s(wng)\, dn$$

where $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Without loss of generality we can take $g = a \in A$, and then by conjugating a across n and w (with suitable change of measure on N) we see that this integral is a scalar multiple of a^{1-s}. (The fact that this scalar has an Euler product, etc., is not of concern at the moment, so we need not do anything further for now.) ♣

Fix a large positive real number ℓ and let

$$L^2(\Gamma\backslash G/K, \ell) = \{\ f \text{ locally integrable on } \Gamma\backslash G/K \text{ so that } \int_{\Gamma\backslash G} |f(x)|^2\, a_x^{-2\ell}\, dx < +\infty\}$$

be a fixed Hilbert space of ℓ-moderate-growth right K-invariant functions on $\Gamma\backslash G$. By the *theory of the constant term*, a convergent Eisenstein series E_s lies in $L^2(\Gamma\backslash G/K, \ell)$ if ℓ is large enough.

Let η be a K-conjugation invariant test function on G. As usual we let such η act on any (quasi-complete, locally convex) G-representation space V by

$$\eta \cdot v = \int_G \eta(g)\, g \cdot v\, dg$$

Since $v \to \eta v$ is a K-morphism, K-isotypes are preserved by η. Then, since the (principal series) representation V_s generated by $g \to a_g^s$ has a one-dimensional K-fixed subspace, for any conjugation-invariant test function η

$$\eta \cdot a^s = \lambda_s \cdot a^s$$

for a constant λ_s depending upon s. Since formation of Eisenstein series (in the region of convergence) is a sum of *left* translates of a^s, in that region we have

$$\eta \cdot E_s = \lambda_s \cdot E_s$$

Lemma: For any η, the corresponding function $s \to \lambda_s$ is *entire* in $s \in \mathbf{C}$. We can choose η so that λ_s is *non-constant*, and in particular not identically zero. Further, given s_o, there is a choice of η so that $\lambda_{s_o} \neq 0$.

Proof: (omitted for now) ♣

Consider the holomorphically parametrized system X of equations in the space $L^2(\Gamma\backslash G/K, \ell)$:

$$\begin{array}{rcll} (a\frac{\partial}{\partial a} - (1-s)) \cdot c_P v_s & = & (2s-1) \cdot a^s & \\ (\eta - \lambda_s) \cdot v_s & = & 0 & \text{for all } \eta \text{ in } C_c^\infty(G)^{\text{inv}} \end{array}$$

The first equation involves a map to *distributions* on A, since we have no assurance that $c_P v_s$ is differentiable.

Remark: The second condition is chosen so that the operator annihilates the more mysterious summand of the two summand in the constant term. We must not presume anything in advance about that more mysterious summand, since in fact we wish to use the theory of Eisenstein series to *prove* things about such entities.

Theorem: The Eisenstein series E_s has a meromorphic continuation to $s \in \mathbf{C}$. Moreover, there exists a meromorphic $\mathbf{C}$-valued function $c(s)$ such that $E(1-s) = c(1-s)E_s$ and $c(s)c(1-s) = 1$, and

$$c_P E_s = a^s + c(s)\, a^{1-s}$$

Proof: First, we prove *uniqueness*: the Eisenstein series E_s is the unique solution to X_s for s in some non-empty open set.

For $\mathrm{Re}\,(s)$ large positive the system X_s has at least the solution E_s, since for large $\mathrm{Re}\,(s)$ the series for E_s converges uniformly on compacta. Suppose that for some fixed s with $\mathrm{Re}\,(s)$ large positive the system X_s has two distinct solutions v_1 and v_2. Then their difference will satisfy

$$\left\{ \begin{array}{rcl} (\eta - \lambda_s)(v_1 - v_2) & = & 0 \\ c_P(v_1 - v_2) & = & b \cdot y^{1-s} \end{array} \right.$$

for some complex number b, where for each $\eta \in C_c^\infty(G)^{\text{inv}}$ the eigenvalue of φ_s is λ_s. By the **'theory of the constant term'**, any left $(\Gamma \cap N)$-invariant function f of moderate growth on $(N \cap \Gamma)\backslash G$ with $\eta f = f$ for some $\eta \in C_c^\infty(G)^{\text{inv}}$ has the property that $f(x) - c_P f(x)$ is of *rapid decay* as $a_x \to +\infty$ (in Siegel sets). Therefore, on Siegel sets the difference $v_1 - v_2$ is

$$v_1 - v_2 = by^{1-s} + (\text{rapid decay})$$

Hence $v_1 - v_2$ is in $L^2(\Gamma\backslash G/K, \ell) \cap L^2(\Gamma\backslash G)$.

Choose a real-valued test function η in $C_c^\infty(()^{\text{inv}}G)$ such that $\eta(g) = \eta(g^{-1})$ for all g. The corresponding operator R_η in $L^2(\Gamma\backslash G)$ is bounded, and, by direct computation of its Hilbert-space adjoint, it is *self-adjoint*. Further, refine the choice of η so that λ_s is not constant, and is non-zero on a neighborhood of s_o with $\mathrm{Re}\,(s_o)$ large. For complex s in a neighborhood of s_o, using the inner product $\langle , \rangle$ on $L^2(\Gamma\backslash G)$, we have

$$\lambda_s \cdot \langle f, f\rangle = \langle R_\eta f, f\rangle = \langle f, R_\eta f\rangle = \overline{\lambda_s} \cdot \langle f, f\rangle$$

This proves that either λ_s is real or $\langle f, f\rangle = 0$. Of course the latter holds only if $f = 0$. But by elementary properties of holomorphic functions, on any non-empty open the non-constant holomorphic function λ_s takes on *some* non-real values. This proves that $f = 0$, and proves the uniqueness.

Now we prove **finiteness**. Fix a **cutoff point** t_o for sufficiently large real t_o. For a function f on $N\backslash G/K$ define its **tail** (above t_o) to be

$$\text{tail}\,(f)(x) = \left\{ \begin{array}{cl} f(x) & \text{if } a_x^1 > t_o \\ 0 & \text{else} \end{array} \right.$$

Define a **truncation operator**

$$\text{trunc } : L^2(\Gamma\backslash G/K, \ell) \to L^2(\Gamma\backslash G/K, \ell)$$

by

$$\text{trunc } (f) = f - E(\text{tail } (c_P f))$$

Then trunc (f) differs from f only in points $x \in G$ with $a_x^1 > t_o$, by subtracting from $f(x)$ its average over $(\Gamma \cap N)\backslash N$. Note that $L^2(\Gamma\backslash G/K, \ell)$ is trunc -stable.

Theorem: (Selberg, Gelfand, Piatetskii-Shapiro) Fix $\eta \in C_c^\infty(K\backslash G)$. For $f \in L^2(\Gamma\backslash G/K, \ell)$ the function $(\eta \circ \text{trunc })f$ is smooth, and all its derivatives are rapidly decreasing functions on Siegel sets. Further, the operator $\eta \circ \text{trunc } : L^2(\Gamma\backslash G/K, \ell) \to L^2(\Gamma\backslash G/K, \ell)$ is *compact.* ♣

Remark: In fact, what is true is that on any Hilbert space of functions of *sufficiently strong polynomial decay* the operator η is compact.

Now we can prove that X has a finite holomorphic envelope (is of finite type). Fix s_o and prove that X_s has a finite envelope locally in a neighborhood of s_o. Let $v \in L^2(\Gamma\backslash G/K, \ell)$ be a solution of X_s for some s close to s_o. Then $c_P v$ is of the form $c_1 a^s + c_2 a^{1-s}$ (for s away from finitely-many points) for some constants c_1, c_2 possibly depending upon s, by the observation that a^s and a^{1-s} are two linearly independent solutions of the differential equation required of the constant term.

From the *theory of the constant term*, since v satisfies $\eta v = v$ for some $\eta \in C_c^\infty(G)^{\text{inv}}$ and is of moderate growth on Siegel sets,

$$v = E(\text{tail } (c_1 a^s + c_2 a^{1-s})) + (\text{rapidly decreasing})$$

In particular, $v \in L^2(\Gamma\backslash G/K, \ell)$ with ℓ chosen as above depending on the range of s. Put

$$L_a^2(\Gamma\backslash G/K, \ell) = \{f \in V_\ell : \text{trunc } \ f = f\}$$

and consider a new space $V' = \mathbf{C} \oplus \mathbf{C} \oplus L_a^2(\Gamma\backslash G/K, \ell)$ with the natural topological vectorspace structure. (The finite dimensional $\mathbf{C}^2$ has a unique topology, and the direct sum has a unique topology whose restriction to $V_{\ell,o}$ is the original. Thus, there is no *choice* of topology.) We define a family of continuous linear maps $T_s : V' \to L^2(\Gamma\backslash G/K, \ell)$ by

$$T_s(b, c, h) = E(\text{tail } (ba^s + ca^{1-s})) + h$$

We must show that $s \to T_s$ is holomorphic in the uniform-norm operator topology on operators between these two Hilbert spaces. The restriction to $L_a^2(\Gamma\backslash G/K, \ell)$ is just the inclusion map, and does not depend upon s, so is certainly holomorphic. Thus, it suffices to check that the family of linear maps $\mathbf{C}^2 \to L^2(\Gamma\backslash G/K, \ell)$ defined by

$$(b, c) \to E(\text{tail } (ba^s + ca^{1-s}))$$

is holomorphic in s. But, since $\varphi \to E(\varphi)$ is linear, it suffices to prove holomorphy of the $L^2(\Gamma\backslash G/K, \ell)$-valued function

$$s \to E(\text{tail } (a^s))$$

(which is entire in s). This is a direct computation.

Next, consider the homogeneous system X'_s given via a single equation $T'_s(v') = 0$, where

$$T'_s : V' \to L^2(\Gamma\backslash G/K, \ell)$$

is

$$T'_s = (\eta - \lambda_s) \circ T_s$$

Note that because λ_s is a holomorphic scalar-valued function, and because η does not vary with s, the parametrized family of operators

$$s \to \eta - \lambda_s$$

is holomorphic in the uniform-norm topology on operators. It is enough to check that X' is of finite type. By the compact-operator criterion it is enough to check that T'_{s_o} has a left inverse A modulo compact operators. Define $A : L^2(\Gamma\backslash G/K, \ell) \to V'$ by

$$Av = (0, 0, \text{trunc } v)$$

On the subspace $L^2_a(\Gamma\backslash G/K, \ell)$ the operator $A \circ T'_{s_o}$ is given by

$$A \circ T'_{s_o}(h) = \eta_{s_o}\text{trunc } (h) - \lambda_{s_o}\text{trunc } (h) = \eta h - \lambda_{s_o}\text{trunc } (h)$$

That is, it differs from the non-zero scalar λ_{s_o} operator by the compact operator $\eta \circ \text{trunc}$. Since the space $L^2_a(\Gamma\backslash G/K, \ell)$ has finite codimension in V',

$$A \circ T_{s_o} = -\lambda_{s_o} + \text{ (compact operator)}$$

which allows use of the compact-operator criterion. This finishes the proof of finiteness.

Since we have uniqueness and finiteness we can apply the continuation principle: the system X_s has a unique solution v_s for s off a locally finite set of analytic hypersurfaces and $s \to v_s$ is meromorphic in s. This **proves the meromorphic continuation of the Eisenstein series.**

Further, we can obtain the functional equation satisfied by the constant term and by the Eisenstein series, as in the statement of the theorem. Define $c(s)$ by

$$c_P E_s = y^s + c(s) y^{1-s}$$

More precisely,

$$(y\frac{\partial}{\partial y} - s)c_P E_s = (1-2s) \cdot c(s) \cdot y^{1-s}$$

Then $c(s)$ has a meromorphic continuation since E_s has. Divide by $c(s)$ and replace s by $1-s$:

$$(y\frac{\partial}{\partial y} - (1-s))c_P E(1-s)/c(1-s) = (2s-1) \cdot c(1-s) \cdot y^s$$

Thus, the constant-term part of the parametrized system defining E_s is also satisfied by $E(1-s)/c(1-s)$.

Lemma: For $\eta \in C_c^\infty(G)^{\text{inv}}$, with λ_s defined by

$$\eta\varphi_s = \lambda_s$$

we have the relation

$$\lambda_{1-s} = \lambda_s$$

Proof: This results from the existence of the G-intertwining map

$$f \to \left(x \to \int_N f(wnx)\, dn\right)$$

from the principal series at s, in which φ_s is the spherical vector, to the principal series at $1-s$, in which φ_{1-s} is the spherical vector, for non-trivial Weyl element w. ♣

Thus, by the uniqueness we have the equality (**functional equation** of the Eisenstein series)

$$E_s = E(1-s)/c(1-s)$$

By replacing s by $1-s$, we have $E(1-s) = E(s)/c(s)$ and hence $c(1-s) = c(s)^{-1}$. ♣