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Representations of moderate growth

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Representations of reductive real Lie groups on Banach spaces, and on the *smooth vectors* in Banach space representations, are *of moderate growth* in a natural sense (below). Among other applications, this allows integrals of the form

$$\int_G f(g)\, g \cdot v \, dg$$

for $v \in V$ with functions f on G in a wider class than merely compactly supported and continuous. For f in the restricted class of compactly supported and continuous, it is relatively elementary to verify that such integrals exist when V is merely (locally convex and) *quasi-complete* (also called *locally complete*). However, in applications to analytic vectors and other things it is necessary to allow integrals which are not compactly supported. An adelic version of this notion of norm also arises in reduction theory, and is useful in discussion of Arthur's truncation operators. We more-or-less follow [Wallach 1982], with some amplification and additions.

The proof that Banach representations are of moderate growth is merely a simpler form of the argument for Fréchet spaces, but is worth giving for clarity. Similarly, the argument that all norms are inessentially different resembles the Banach-space argument for moderate growth.

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1. Constructing norms on groups

Let G be a linear reductive group, a closed subgroup of $GL(n, \mathbf{R})$, with maximal compact subgroup K.

Definition: A **norm** [1] $g \to |g|$ on G is a continuous non-negative real-valued function such that
• $|g| \geq 1$
• $|g^{-1}| = |g|$
• $|gh| \leq |g||h|$
• Given a bound $r > 0$, $\{g \in G : |g| \leq r\}$ is compact.
• For $g \in G$, $k \in K$

$$|k \cdot g| = |g \cdot k| = |g|$$

• For $X \in \mathbf{p}$, for $t \geq 0$,

$$|e^{tX}| = |e^X|^t$$

We construct one such norm now, and later show that any other norm differs in inessential ways from the one constructed.

[1] This norm is not a norm in the Banach-space sense, since it will not be positive-homogeneous in $g \in G$.

Suppose we have an involution [2] σ on G so that K is the fixed-point set of σ on G. We can alter either the choice of K or the choice of imbedding so that σ is the restriction to (the image of) G of *transpose-inverse* $g \to (g^\top)^{-1}$ on $GL(n, \mathbf{R})$. Thus, $K \subset O(n)$ with the standard orthogonal group $O(n) \subset GL(n, \mathbf{R})$. The Lie algebra $\mathbf{k}$ of K is also the $(+1)$-eigenspace of the induced action $\gamma \to -\gamma^\top$ of σ on the Lie algebra $\mathbf{g}$ of G, and let $\mathbf{p}$ be the $(+1)$-eigenspace of σ on $\mathbf{g}$.

Imbed G in $GL(2n, \mathbf{R})$ by

$$g \to \begin{pmatrix} g & 0 \\ 0 & (g^\top)^{-1} \end{pmatrix}$$

Give $\mathbf{R}^{2n}$ a Hilbert space structure $\langle , \rangle$, which we can assume to be K-invariant in the sense

$$\langle k \cdot u, k \cdot v \rangle = \langle u, v \rangle$$

for all $u, v \in \mathbf{R}^{2n}$ and $k \in K$ acting via the previous imbedding of G in $GL(2n, \mathbf{R})$. Let

$$|x| = \langle x, x \rangle^{1/2}$$

be the norm attached to the inner product. On any real or complex Banach space V there is the uniform operator norm

$$|h|_{V,\mathrm{op}} = \sup_{x \in V,\ |x| \le 1} |hx|$$

Definition: The **norm** [3] on $G \subset GL(n, \mathbf{R})$ is

$$|g| = \left| \begin{pmatrix} g & 0 \\ 0 & (g^\top)^{-1} \end{pmatrix} \right|_{\mathbf{R}^{2n},\mathrm{op}} = \max(|g|_{\mathbf{R}^n,\mathrm{op}},\ |(g^\top)^{-1}|_{\mathbf{R}^n,\mathrm{op}}) = \max(|g|_{\mathbf{R}^n,\mathrm{op}},\ |g^{-1}|_{\mathbf{R}^n,\mathrm{op}})$$

where any $\mathbf{R}^n$ is given the usual Hilbert-space structure.

Proposition: The function $g \to |g|$ explicitly defined just above is a *norm* in the sense axiomatized earlier.

Proof: This is straightforward, and it certainly suffices to think of $G = GL(n, \mathbf{R})$ for all these assertions, since we have aligned the involution on G to match transpose-inverse. For G acting on any Banach space V we have

$$1 = |1_V|_{V,\mathrm{op}} = |g \circ g^{-1}|_{V,\mathrm{op}} \le |g|_{V,\mathrm{op}} \cdot |g^{-1}|_{V,\mathrm{op}}$$

From this, either g or g^{-1} has operator norm at least 1. Thus, $|g| \ge 1$. Likewise, by definition, $|g^{-1}| = |g|$.

The K-invariance follows from our arranging K to be inside $O(n)$, which preserves the usual Hilbert space structure on $\mathbf{R}^n$.

For the submultiplicativity, let $\| * \| = | * |_{\mathbf{R}^n, op}$ and note that

$$|gh| = \max(\|gh\|,\ \|(gh)^{-1}\|) \le \max(\|g\| \cdot \|h\|,\ \|h^{-1}\| \cdot \|g^{-1}\|) \le \max(\|g\|, \|g^{-1}\|) \cdot \max(\|h\|, \|h^{-1})) = |g| \cdot |h|$$

Since G is a closed subgroup of $GL(n, \mathbf{R})$, to prove compactness of the subset B_r of G on which $|g| \le r$ it suffices to prove this closedness for $G = GL(n, \mathbf{R})$. One should prove the auxiliary result that

$$g \to \begin{pmatrix} g & 0 \\ 0 & (g^\top)^{-1} \end{pmatrix}$$

[2] For us an *involution* will be an automorphism whose square is the identity map. Thus, $(gh)^\sigma = g^\sigma h^\sigma$, rather than application of σ reversing the order.

[3] This apparently depends upon the imbedding of G in $GL(n, \mathbf{R})$.

imbeds $GL(n, \mathbf{R})$ as a *closed* subset of the space of $2n$-by-$2n$ real matrices. For S, T in that space of matrices, it is a standard result that the operator norm $|S - T|_{\mathbf{R}^{2n},\mathrm{op}}$ is a (complete) metric.

Finally, let X be a symmetric n-by-n real matrix. By the spectral theorem for finite-dimensional symmetric operators, $X = k\delta k^{-1} = k\delta k^\top$ for some real diagonal δ and $k \in O(n)$. Since the norm is $O(n)$-invariant, it suffices to treat $X = \delta$. Let δ_i be the diagonal entries of δ. Since the operator norm of a matrix is the sup of the absolute values of its entries, for $t > 0$

$$|e^{tX}| = \sup_i \max(e^{t\delta_i}, e^{-t\delta_i}) = \sup_i e^{t|\delta_i|} = (\sup_i e^{|\delta_i|})^t$$

This proves the last property. ///

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2. Banach representations are of moderate growth

As usual, a *representation* of G on a topological vectorspace V is assumed *continuous* in the sense that the map

$$G \times V \to V$$

by $(g, v) \to g \cdot v$ is a continuous V-valued function on $G \times V$.

Definition: A representation of G on a Banach space V is **of moderate growth** if, given a norm $g \to |g|$ on G, there is a constant C and an exponent M such that for all $g \in G$ and for all $v \in V$

$$|g \cdot v|_V \leq |v|_V \cdot |g|^M$$

where $| * |_V$ is the norm on V.

Theorem: A Banach-space representation V of a linear reductive real Lie group G is of moderate growth. [4]

Proof: For brevity, let

$$\|g\| = |g|_{V,\mathrm{op}}$$

Claim that the operator norm $\|g\|$ is *bounded* on the compact set

$$B_r = \{g \in G : |g| \leq r\} \subset G$$

for any bound r where $|g|$ is the norm on G. Indeed, B_r is compact and $g \to g \cdot v$ is continuous for each v, so the Uniform Boundedness Theorem [5] gives the claim.

Let $C > 0$ be a bound for $\|g\|$ on B_3. We claim that

$$\frac{1}{C}\|g\| \leq \|kg\| \leq C\|g\|$$

[4] For $G \times V \to V$ a representation on a general topological vector space V whose topology is given by seminorms μ, if G acts continuously on each completion (in a generalized sense) V_μ of V with respect to μ, then the same proof shows that the action of G on V is of moderate growth in an obvious sense. However, not all continuous actions on topological vector spaces meet this criterion. The important case of *smooth vectors* in a representation already know to be of moderate growth, treated below, allows a special argument.

[5] *Uniform Boundedness Theorem* often refers to a simple Banach-space instance of the more general *Banach-Steinhaus* theorem. A sufficient and relatively simple version for Banach spaces is the following. Let X and Y be Banach spaces, and Φ a set of continuous linear maps $X \to Y$. If for each $\varphi \in \Phi$ the set $\{|\varphi(x)| : |x| \leq 1\}$ is *bounded*, then there is a single *uniform* bound for all $\varphi \in \Phi$. That is, there is $B < \infty$ such that $|\varphi(x)| \leq B$ for all $|x| \leq 1$ and $\varphi \in \Phi$.

for all $g \in G$, $k \in K$. Since $K \subset B_r$ for $r > 1$, we have $\|k\| \le C$ for $k \in K$. Then the submultiplicativity of $\| * \|$ gives one part of the inequality. On the other hand,

$$\|g\| = \|k^{-1}kg\| \le \|k^{-1}\| \cdot \|kg\| \le C\,|kg\|$$

which yields the other inequality upon replacing g by $k^{-1}g$ and k by k^{-1}. This gives the claim.

Next, claim that for $X \in \mathbf{p}$

$$\|e^X\| \le C \cdot |e^X|^C$$

where C is the sup of $\|g\|$ for $|g| \le 3$. To verify this, let ℓ be the least positive integer such that

$$|e^X| \le e^\ell$$

Then by the property $|e^{tX}| = |e^X|^t$ for $t > 0$, proven earlier, taking $t = \ell^{-1}$ gives

$$|e^{\ell^{-1}X}| = |e^X|^{\ell^{-1}} \le (e^\ell)^{\ell^{-1}} \le e < 3$$

Therefore, since $\|g\| \le C$ for $|g| \le 3$,

$$\|e^{\ell^{-1}X}\| \le C$$

and

$$\|e^X\| = \|e^{\ell\cdot\ell^{-1}X}\| \le \|e^{\ell^{-1}X}\|^\ell \le C^\ell \le C^{1+\log|e^X|} = C \cdot |e^X|^C$$

Now we give the main estimate. For $k \in K$ (again noting $|k| = 1 < 3$)

$$\|k\,e^X\| \le \|k\| \cdot \|e^X\| \le C \cdot (C \cdot |e^X|^C) = C^2 \cdot (1 \cdot |e^X|^C) = C^2 \cdot |ke^X|^C$$

since $|kg| = |g|$. By the Cartan decomposition $G = K \cdot (\exp \mathbf{p})$, we obtain the inequality $\|g\| \le C^2\,|g|^C$ of the theorem for all elements of the group. ///

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3. Essential equivalence of norms on reductive groups

An argument very similar to that for the moderate growth of Banach space representations will prove that any two norms differ in inessential ways. Thus, the notion of moderate growth is intrinsic.

Proposition: Let $| * |$ and $\| * \|$ be norms on G. Then

$$\|g\| \le C \cdot |g|^M$$

for all $g \in G$, for some constants C and M.

Proof: Claim that $\|g\|$ is *bounded* on the compact set

$$B_r = \{g \in G : |g| \le r\} \subset G$$

for any bound r where $|g|$ is the norm on G. Indeed, B_r is compact and $g \to \|g\|$ is continuous.

Let $C > 0$ be a bound for $\|g\|$ on B_3. We claim that

$$\frac{1}{C}\|g\| \le \|kg\| \le C\|g\|$$

for all $g \in G$, $k \in K$. Since $K \subset B_r$ for $r > 1$, we have $\|k\| \le C$ for $k \in K$. Then the submultiplicativity of the norm $\| * \|$ gives one part of the inequality. On the other hand,

$$\|g\| = \|k^{-1}kg\| \le \|k^{-1}\| \cdot \|kg\| \le C\,|kg\|$$

which yields the other inequality upon replacing g by $k^{-1}g$ and k by k^{-1}. This gives the claim.

Next, claim that for $X \in \mathbf{p}$

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where C is the sup of $\|g\|$ for $|g| \le 3$. To verify this, let ℓ be the least positive integer such that

$$|e^X| \le e^\ell$$

Then by the property $|e^{tX}| = |e^X|^t$ for $t > 0$, proven earlier, taking $t = \ell^{-1}$ gives

$$|e^{\ell^{-1}X}| = |e^X|^{\ell^{-1}} \le (e^\ell)^{\ell^{-1}} \le e < 3$$

Therefore, since $\|g\| \le C$ for $|g| \le 3$,

$$\|e^{\ell^{-1}X}\| \le C$$

and

$$\|e^X\| = \|e^{\ell \cdot \ell^{-1}X}\| \le \|e^{\ell^{-1}X}\|^\ell \le C^\ell \le C^{1+\log|e^X|} = C \cdot |e^X|^C$$

Now the main estimate. For $k \in K$

$$\|k\,e^X\| \le \|k\| \cdot \|e^X\| \le C \cdot (C \cdot |e^X|^C) = C^2 \cdot (1 \cdot |e^X|^C) = C^2 \cdot |ke^X|^C$$

since $|kg| = |g|$. By $G = K \cdot (\exp \mathbf{p})$, we obtain the inequality $\|g\| \le C^2\,|g|^C$ of the theorem for all elements of the group. ///

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4. Smooth vectors

Let $G \times V \to V$ be a representation of G on a locally convex topological vector space V. As usual, a vector $v \in V$ is **smooth** if

$$g \to g \cdot v$$

is a smooth V-valued function on G. The subspace of smooth vectors in V is denoted V^∞. Often, the space of smooth vectors in a representation is of as great interest as the original representation. [6] For X in the universal enveloping algebra $U(\mathbf{g})$ of the Lie algebra $\mathbf{g}$ of G, and for a seminorm μ on V, define another seminorm on $v \in V^\infty$ by

$$\mu_X(v) = \mu(X \cdot v)$$

where the action of the universal enveloping algebra is defined on smooth vectors as usual by taking the extension of the natural action

$$Xf(g) = \frac{d}{dt}\,|_{t=0}\,\left(e^{tX} \cdot v\right) \qquad \text{(for } X \in \mathbf{g}\text{)}$$

[6] For V locally convex and quasi-complete (bounded Cauchy *nets* converge) one has a good theory of Gelfand-Pettis (weak) integrals $\varphi \cdot v = \int_G \varphi(g)\, g \cdot v\, dg$ for compactly supported smooth φ on G. Such vectors are unsurprisingly demonstrably smooth, and consideration of φ in an *approximate identity* yields the density of the smooth vectors V^∞ in such V.

Proposition: For $G \times V \to V$ a (continuous) representation of G with locally convex V then so is the associated $G \times V^\infty \to V^\infty$, with the locally convex topology just described, is also a (continuous) representation. [7]

Proof: Continuity is the issue. Let $U^{\leq n}(\mathbf{g})$ be the part of $U(\mathbf{g})$ consisting of images of tensors of degrees $\leq n$. [8] The topology given by the seminorms μ_X by design makes all the maps

$$v \to X \cdot v$$

continuous, for fixed $X \in U(\mathbf{g})$. Since each $U^{\leq n}\mathbf{g})$ is finite-dimensional, all the maps

$$X \to X \cdot v$$

for fixed $v \in V^\infty$ are inevitably continuous. That is, the bilinear map

$$U^{\leq n}(\mathbf{g}) \times V^\infty \to V^\infty$$

is *separately* continuous. We need to prove that it is *jointly* continuous.

Since $U^{\leq n}\mathbf{g})$ is finite-dimensional it is locally compact, so the image of the closed unit ball B (with any Banach space structure) in V^∞ under any map $X \to Xv$ is compact. Thus, by Banach-Steinhaus [9] the collection of operators on V^∞ coming from $X \in B$ is *equicontinuous*. Thus, the bilinear map

$$U^{\leq n}(\mathbf{g}) \times V^\infty \to V^\infty$$

is *jointly* continuous. And, since the Adjoint action of G on $U(\mathbf{g})$ preserves the degree filtration on $U(\mathbf{g})$, and since $U^{\leq n}(\mathbf{g})$ is finite-dimensional, this action of G on $U(\mathbf{g})$ is necessarily continuous.

For fixed $X \in U \leq n(\mathbf{g})$, and for seminorm μ on V, let $v, w \in V$ and $g, h \in G$, and consider

$$\mu_X(g \cdot v - h \cdot w) = \mu(X \cdot gv - X \cdot hw) = \mu(g \cdot (\mathrm{Ad}g^{-1})(X) \cdot v - h \cdot (\mathrm{Ad}h^{-1})(X) \cdot w)$$

For h close to g, $(\mathrm{Ad}g^{-1})X$ is close to $(\mathrm{Ad}h^{-1})X$, so (by the joint continuity of $U^{\leq n}(\mathbf{g}) \times V^\infty \to V^\infty$) for v close to w in the V^∞ topology, $(\mathrm{Ad}g^{-1})(X) \cdot v$ is close to $h \cdot (\mathrm{Ad}h^{-1})(X) \cdot w$ in the V^∞ topology. The (joint) continuity of $G \times V \to V$ with the original topology on V then shows that

$$\mu(g \cdot (\mathrm{Ad}g^{-1})(X) \cdot vh \cdot (\mathrm{Ad}h^{-1})(X) \cdot w)$$

is small for h close to g and w close to v. This proves that $G \times V^\infty \to V^\infty$ is continuous. ///

[7] Further, though we don't need it here, it is true that for locally convex *quasi-complete* V, the smooth vectors V^∞ are also quasi-complete. (As usual, *quasi-completeness* means that bounded (in the topological vector space sense) Cauchy *nets* converge. This implies *sequential* completeness. Many dual spaces with weak $*$-topologies are quasi-complete, but rarely complete in the strongest sense that *all* Cauchy nets are convergent.)

[8] Since $U(\mathbf{g})$ is a quotient of the tensor algebra of $\mathbf{g}$ by an ideal which is *not* homogeneous, we do not have a *grading* of $U(\mathbf{g})$, but only a *filtration*.

[9] A more general version of Banach-Steinhaus is the following. Recall that a subset E of a topological vector space Y is *bounded* if, for any neighborhood U of 0 in Y, there is sufficiently large real t_o such that for all $t \geq t_o$ we have $tN \supset E$. In *any* topological vector space Y, *compact* sets are bounded. A set of linear maps Φ from one topological vector space X to another Y is *equicontinuous* if given a neighborhood U of 0 in Y there is a neighborhood N of 0 in X such that $\varphi(N) \subset U$ for all $\varphi \in \Phi$. Let Φ be a collection of continuous linear maps $X \to Y$ from a Fréchet space X to a topological vector space Y, such that for every $x \in X$ the image $\{\varphi(x) : \varphi \in \Phi\}$ is *bounded* in Y. Then the Banach-Steinhaus theorem asserts that Φ is *equicontinuous*.

5. Moderate growth of smooth vectors

Here we show that in a moderate-growth representation the smooth vectors are also of moderate growth. We extend the definition of *moderate growth* representations:

Definition: A representation of G on a locally convex topological vectorspace V is **of moderate growth** if, given a norm $g \to |g|$ on G, for every continuous seminorm μ on V there is a seminorm ν on V and an exponent M such that for all $g \in G$ and for all $v \in V$

$$\mu(g \cdot v) \le \nu(v) \cdot |g|^M$$

Theorem: Let $G \times V \to V$ be a moderate-growth representation of a reductive real Lie group G on a locally convex space V. Then the subrepresentation V^∞ of smooth vectors, with the topology given above, is also of moderate growth.

Proof: Fix $v \in V^\infty$ and $X \in U(\mathbf{g})$. By the moderate growth of the original representation, for some seminorm ν on V, and for some exponent M,

$$\mu_X(g \cdot v) = \mu(g \cdot (\mathrm{Ad}g^{-1})X \cdot v) \le \nu((\mathrm{Ad}g^{-1})X \cdot v) \cdot |g|^M$$

Let n be an integer such that X lies in the degree $\le n$ filtered part $U^{\le n}(\mathbf{g})$ of $U(\mathbf{g})$, and give this filtered part a norm [10] $\| * \|$. The moderate growth of the Adjoint action of G on $U^{\le n}(\mathbf{g})$ gives

$$\|(\mathrm{Ad}g^{-1})X\| \le \|X\| \cdot C\, |g^{-1}|^N = \|X\| \cdot C\, |g^{-1}|^N$$

for some constant C and exponent N. The joint continuity $U^{\le n}(\mathbf{g})$ proven in the course of the proof in the last section asserts, in terms of seminorms, that given a seminorm p on V^∞ there exists a seminorm q on V^∞ such that, for $Y \in U^{\le n}(\mathbf{g})$,

$$p(Y \cdot v) \le \|Y\| \cdot q(v)$$

Thus, for some seminorm q on V^∞,

$$\nu((\mathrm{Ad}g^{-1})X \cdot v) \le \|(\mathrm{Ad}g^{-1})X\| \cdot q(v) \le C \cdot |g|^N \cdot \|X\| \cdot q(v)$$

Putting this together,

$$\mu_X(g \cdot v) = \mu(g \cdot (\mathrm{Ad}g^{-1})X \cdot v) \le \nu((\mathrm{Ad}g^{-1})X \cdot v) \cdot |g|^M \le C \cdot |g|^{M+N} \cdot \|X\| \cdot q(v)$$

The constant $C \cdot \|X\|$ can be absorbed into the seminorm q on $v \in V^\infty$, so we have proven the moderate growth of V^∞. ///

6. Polynomial growth of reductive groups

Next, we show that modifying Haar measure on G by any sufficiently negative power of a norm gives G finite total mass. [11]

[10] Since $U^{\le n}(\mathbf{g})$ is finite-dimensional, all choices of norm give idential topologies.

[11] It is enough that G be reductive, not necessarily semi-simple, since the center of a reductive G acts non-trivially in the standard representation associated to the imbedding of G in $GL(n, \mathbf{R})$. This is indeed in contrast to the *Ad* action of a reductive G on its own Lie algebra, under which the center of G does nothing.

Let Φ^+ be the collection of positive roots in $\mathbf{g}$, with respect to a fixed maximal $\mathbf{R}$-split torus A in G, and let $\mathbf{g}_\alpha$ be the α^{th} root space. Let A^+ be the connected component of the identity in A. For a character λ on A^+, it is convenient to let

$$a \to a^\lambda$$

denote the value of λ on $a \in A^+$.

Lemma: With the norm constructed above, given a character λ on A^+, there is a constant C such that

$$a^\lambda \le |a|^C$$

Proof: Let Φ be the collection of weights of the Lie algebra $\mathbf{a}$ of A^+ on $\mathbf{R}^{2n}$ on which $G \subset GL(2n, \mathbf{R})$ is made to act via the imbedding $g \to g \oplus (g^\top)^{-1}$. Let $\| * \| = | * |_{\mathbf{R}^{2n},\mathrm{op}}$. By the nature of the imbedding Φ spans the dual of $\mathbf{a}$. For any $a \in A$

$$|a| = \|a\| + \|(a^\top)^{-1}\| = \|a\| + \|a^{-1}\| = \max_{\alpha \in \Phi} a^\alpha$$

since, by the nature of the imbedding, if $a \to a^\alpha$ is in Φ, then $a \to a^{-\alpha}$ is also in Φ. Since Φ spans the dual space of $\mathbf{a}$ and is closed under -1, any weight λ of $\mathbf{a}$ is a linear combination $\lambda = \sum_{\alpha \in \Phi} c_\alpha\, \alpha$ of elements of Φ with coefficients $c_\alpha \ge 0$. Let $C = \sum_\alpha c_\alpha$. Then

$$a^\lambda = a^{\Sigma c_\alpha\, \alpha} = \prod_\alpha (a^\alpha)^{c_\alpha} \le \prod_\alpha |a|^{c_\alpha} = |a|^C$$

as claimed. ///

Define as usual, for $a \in A^+$,

$$D(a) = \prod_{\alpha \in \Phi^+} |\sinh(a^\alpha)|^{\dim \mathbf{g}_\alpha}$$

The (Weyl) integration formula is standard:

Lemma:

$$\int_G f(g)\, dg = \int_K \int_K \int_{A^+} f(k_1 a k_2)\, D(a)\, dk_1\, dk_2\, da$$

for compactly supported continuous f. ///

Proposition: With the norm $| * |$ on G as constructed above, for sufficiently large positive exponent M

$$\int_G |g|^{-M}\, dg < \infty$$

Proof: For each $\alpha \in \Phi^+$, let C_α be the exponent (from the next to last lemma) such that $a^\alpha \le |a|^{C_\alpha}$. Then certainly

$$\left| \frac{a^\alpha - a^{-\alpha}}{2} \right| \le |a|^{\max(C_\alpha, C_{-\alpha})}$$

Let $S = \sum_{\alpha \in \Phi^+} \max(C_\alpha, C_{-\alpha})$. Thus,

$$D(a) \le \prod_{\alpha \in \Phi^+} |a|^{\max(C_\alpha, C_{-\alpha})} = |a|^S$$

Thus, the issue is reduced to showing that

$$\int_{A^+} |a|^{-t}\, da < +\infty$$

for sufficiently large positive t. This itself reduces to noting the convergence of

$$\int_0^{+\infty} \max(a, a^{-1})^{-t}\, \frac{da}{a}$$

for large positive t, which is elementary. ///

[Wallach 1982], N. Wallach, *Asymptotic expansions of generalized matrix entries of representations of real reductive groups*, in *'Lie Group Rpepresentations I, Proceedings, Maryland, 1982-3'*, SLN 1024, pp. 287-369.