Example computations in automorphic spectral theory

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Desiderata:

• Attention to primordial/fundamental issues...

• ... presumably requiring adaptible, perhaps *extreme* versions of trace formulas or other harmonic analysis of automorphic forms...

• *Robustness*: proofs, not just heuristics...

• *Conceptual* arguments, not cleverness/twigs-and-mud...

(Therefore...) keep things as simple as possible, as long as possible.

[Introductions: ... to automorphic spectral theory, spherical functions: H. Iwaniec' book, Cogdell-PiatetskiShapiro's book, ... to spherical functions in S. Helgason's *Geometric Analysis*, ... Lang-Jorgenson *Heat Kernel...* $SL_2(\mathbb{C})$.]

Automorphic spectral expansions on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$

$$f \;=\; \sum_{\text{cfm } F} \langle f, F\rangle \cdot F \qquad\qquad \text{(cuspidal component)}$$

$$+\; \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} \qquad\qquad \text{(residual spectrum)}$$

$$+\; \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E_s\rangle \cdot E_s\; ds \qquad \text{(continuous)}$$

converge in L^2. But... **locally uniformly pointwise?**

Surely ok ... for sufficiently smooth f, but what does it take to *prove* this? ... to justify manipulation of such expressions as though they had *pointwise* meaning?

Pointwise estimates on cuspforms and Eisenstein series? How hard could it be? **Recall** that for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H} = SL_2(\mathbb{R})/SO(2)$

$$E_s(i) \;=\; 2y^s \cdot \zeta_{\mathbb{Q}(i)}(s)$$

So optimal pointwise estimates for E_s flirt with Lindelöf. Surely cuspforms are subtler. Hard to do this... Alternatives?

Hyperbolic two-space, three-space

$\mathfrak{H} \approx SL_2(\mathbb{R})/SO(2) \approx$ hyperbolic two-space

$SL_2(\mathbb{C})/SU(2) \approx$ hyperbolic three-space

Split component

$$A^+ \;=\; \{a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \;:\; r \geq 0\}$$

Cartan decomposition $G = KA^+K$.

Invariant metric on G/K

$$d(gK, hK) \;=\; r \qquad \text{where} \qquad h^{-1}g \;\in\; Ka_rK$$

$$\text{Haar} \;=\; |\sinh r|\, dk\, dr\, dk' \qquad\qquad (\text{for } SL_2(\mathbb{R}))$$

$$\text{Haar} \;=\; |\sinh r|^2\, dk\, dr\, dk' \qquad\qquad (\text{for } SL_2(\mathbb{C}))$$

Laplacian Δ is Casimir on right K-invariant functions

Eigenvalue $\lambda_s = s(s-1)$ on s^{th} **principal series**

Radial Laplacian on $F(ka_r k') = f(r)$

$$\Delta F \;=\; f'' + \coth r \cdot f' \qquad\qquad (\text{for } SL_2(\mathbb{R}))$$

$$\Delta F \;=\; f'' + 2\coth r \cdot f' \qquad\qquad (\text{for } SL_2(\mathbb{C}))$$

$$\Delta F = f'' + (n-1)\coth r\, f' \qquad (\text{hyperbolic } n\text{-space})$$

Spherical functions: smooth K-bi-invariant Δ-eigenfunctions

Non-trivial general fact: every eigenspace on G/K is image of **unramified principal series**

Convenient elementariness of spherical functions for $SL_2(\mathbb{C})$... for all **complex** groups.

With $f = \varphi/\sinh r$, $\lambda_s = s(s-1)$,

$$\left(\frac{\varphi}{\sinh r}\right)'' + 2\coth r \cdot \left(\frac{\varphi}{\sinh r}\right)'$$

$$= \;\frac{\varphi''}{\sinh r} \;-\; \frac{\varphi}{\sinh r}$$

Eigenvalue equation

$$\frac{\varphi''}{4\sinh r} \;-\; \frac{\varphi}{4\sinh r} \;=\; \lambda_s \cdot \frac{\varphi}{\sinh r}$$

miraculously (?!) becomes constant-coefficient equation

$$\tfrac{1}{4}\big(\varphi'' \ - \ \varphi\big) \ = \ \lambda_s \cdot \varphi$$

With $\varphi(r) = e^{\pm(2s-1)r}$

$$\tfrac{1}{4}\big(\varphi'' \ - \ \varphi\big) \ = \ \lambda_s \cdot e^{\pm(2s-1)r}$$

so

$$\Delta\Big(\frac{e^{\pm(2s-1)r}}{\sinh r}\Big) \ = \ \lambda_s \cdot \frac{e^{\pm(2s-1)r}}{\sinh r}$$

Blow-up at $r \ = \ 0$. Normalize to value 1 at $r = 0$, define s^{th} **spherical function**

$$\varphi_s(r) \ = \ \frac{\sinh(2s-1)r}{(2s-1)\ \sinh r}$$

On unitary line

$$\varphi_{\frac{1}{2}+it}(r) \ = \ \frac{\sin 2tr}{2t\ \sinh r}$$

Spherical transform

$$\widetilde{f}(\tfrac{1}{2}+i\xi) \ = \ \int_G f \cdot \overline{\varphi}_{\frac{1}{2}+i\xi} \ = \ \int_G f \cdot \varphi_{\frac{1}{2}-i\xi}$$

Spherical inversion

$$f \;=\; \int_{-\infty}^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot|\mathbf{c}(\tfrac{1}{2}+i\xi)|^{-2}\,d\xi$$

where $\mathbf{c}(\frac{1}{2}+i\xi) \;=\; \xi^{-1}$. That is,

$$f \;=\; \int_{-\infty}^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot\xi^{2}\,d\xi$$

For $SL_2(\mathbb{C})$, can prove spherical inversion from Fourier inversion on $\mathbb{R}$.

Plancherel

$$\int_G f\cdot F \;=\; \int_0^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\widetilde{F}(\tfrac{1}{2}+i\xi)\cdot\xi^{2}\,d\xi$$

and *pointwise* convergence (...Sobolev...)

$$f(eK) \;=\; \int_{-\infty}^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot 1\cdot\xi^{2}d\xi$$

suggest $\widetilde{\delta}(\frac{1}{2}+i\xi) = 1$. Indeed, ... in spherical **Sobolev space** δ has expansion, **convergent** in that topology,

$$\delta = \int_{-\infty}^{\infty}\widetilde{\delta}(\tfrac{1}{2}+i\xi)\,\varphi_{\frac{1}{2}+i\xi}\,\xi^{2}\,d\xi \;=\; \int_{-\infty}^{\infty}\varphi_{\frac{1}{2}+i\xi}\,\xi^{2}\,d\xi$$

Free space fundamental solution or **Green's function** solution u_z to

$$(\Delta - \lambda_z)^2 \, u_z \; = \; \delta \qquad\qquad (\text{on } G/K)$$

found via spherical transform: spectral decomposition diagonalizes differential operators

$$(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2 \, \widetilde{u}_z(\tfrac{1}{2}+i\xi) \; = \; \widetilde{\delta}(\tfrac{1}{2}+i\xi) \; = \; 1$$

$$\widetilde{u}_z(\tfrac{1}{2}+i\xi) \; = \; \frac{1}{(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2}$$

Spherical inversion

$$u_z \; = \; \int_{-\infty}^{\infty} \varphi_{\frac{1}{2}+i\xi} \, \frac{\xi^2 \, d\xi}{(\lambda_{\frac{1}{2}+i\xi} - \lambda_z)^2}$$

By residues, up to constant,

$$u_z \; = \; \frac{r \, e^{-(2z-1)\, r}}{(2z-1) \, \sinh r}$$

Poincaré series, or **automorphic Green's function**, or **automorphic fundamental solution** for $\Gamma = SL_2(\mathbb{Z}[i])$

$$\mathfrak{P}_z(g,h) \;=\; \sum_{\gamma\in(\Gamma\cap K)\backslash\Gamma} u_z(g^{-1}\gamma h)$$

via *gauges*: converges absolutely, uniformly on compacts in $G \times G$. **As function of** h is in $L^2(\Gamma\backslash G/K)$, for $\mathrm{Re}(z) \gg 1$.

Claim: L^2 **automorphic spectral expansion** in h for $\mathrm{Re}(z) \gg 1$ is

$$\mathfrak{P}_z(g,h) \;=\; \frac{1/\langle 1,1\rangle}{(\lambda_1-\lambda_z)^2} \qquad \text{(residual)}$$

$$+ \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F-\lambda_z)^2} \qquad \text{(cuspidal)}$$

$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)\;ds}{(\lambda_s-\lambda_z)^2} \quad \text{(continuous)}$$

Not attempt to integrate the Poincaré series against Eisenstein series to determine continuous spectrum components.

Rather, use spectral expansion of **automorphic delta** δ^{afc} in **global automorphic Sobolev space**

$$\begin{aligned}\delta_g^{\text{afc}}(h) \;&=\; 1/\langle 1,1\rangle && \text{(residual)}\\ &+ \sum_F F(g)\,\overline{F}(h) && \text{(cuspidal)}\\ &+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_s(g)\,E_{1-s}(h)\,ds && \text{(continuous)}\end{aligned}$$

Converges in suitable Sobolev topology.

Spectral expansions diagonalize invariant differential operators. Solve differential equation

$$(\Delta-\lambda_z)^2 f_z \;=\; \delta^{\text{afc}}$$

by *dividing* by $(\lambda-\lambda_z)^2$ in the spectral expansion, producing *an* automorphic solution f_z of the differential equation.

We have *two* automorphic fundamental solutions of $(\Delta - \lambda_z)^2$: Poincaré series and automorphic-spectrally-obtained solution.

Discussion of automorphic spectral expansions shows there are *no* solutions of the *homogeneous* equation in union $\mathrm{Sob}^{\mathrm{afc}}(-\infty)$ of global automorphic Sobolev spaces.

That is, global automorphic Sobolev theory gives *uniqueness*.

Thus, solution $\mathfrak{P}_z$ wound up from free-space solution is equal to solution obtained by automorphic spectral expansions

That is, the spectral expansion of $\mathfrak{P}_z$ is the obvious thing, written above.

That expansion **converges** not just in $L^2(\Gamma\backslash G/K)$, but also **pointwise**, since

$$\mathrm{Sob}^{\mathrm{afc}}(-(\tfrac{3}{2}+\varepsilon)+4) \;=\; \mathrm{Sob}^{\mathrm{afc}}(\tfrac{5}{2}-\varepsilon)$$

$$\subset\; \mathrm{Sob}^{\mathrm{afc}}(\tfrac{3}{2}+\varepsilon) \;\subset\; C^o(\Gamma\backslash G/K)$$

Standard estimates (to ground global automorphic Sobolev)

$$\sum_{|\lambda_F| \le T} |F(g)|^2 + \frac{1}{2\pi} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} |E_{\frac{1}{2}+it}(g)|^2 \, dt \ll_C T^{\frac{3}{2}}$$

uniformly locally in g. Implies automorphic delta is in $\mathrm{Sob}^{\mathrm{afc}}(-\frac{3}{2} - \varepsilon)$. Implies locally uniform pointwise convergence of $\mathfrak{P}_z$.

Eisenstein series normalized

$$E_{1-s} \;=\; c_{1-s}\, E_s \;=\; \frac{E_s}{c_s}$$

with Fourier expansion

$$\|y\|_{\mathbb{C}}^{s} + c_s \|y\|_{\mathbb{C}}^{1-s} + \ldots \quad = \quad y^{2s} + c_s\, y^{2-2s} + \ldots$$

$\|\cdot\|_{\mathbb{C}}$ is product-formula normalization, and

$$c_s \;=\; \frac{\xi_k(2s-1)}{\xi_k(2s)} \quad (\xi_k(s) \text{ is } \zeta_k(s) \text{ with gamma})$$

In $\mathrm{Re}(z) > \frac{1}{2}$ poles of $\mathfrak{P}_z$ only at $z = 1$ and conceivably at finitely-many $s_F \in (\frac{1}{2}, 1)$. Probably none of the latter, but irrelevant.

Meromorphic continuation to the line $\mathrm{Re}(z) = \frac{1}{2}$ and beyond:

Sum of discrete spectrum components continues with obvious poles.

Continuous spectrum: move z near $\mathrm{Re}(z) = \frac{1}{2}$, deform contour to the right, producing *negative* of a residue,

$$-\mathrm{Res}_{s=z} \frac{E_s \otimes E_{1-s}}{(s-z)^2 \, (s-(1-z))^2}$$

$$= \; -\frac{(E_z \otimes E_{1-z})'}{(2z-1)^2} + 2\frac{E_z \otimes E_{1-z}}{(2z-1)^3}$$

E_z vanishes at $z = \frac{1}{2}$, giving $\mathfrak{P}_z$ simple pole.

To continue to $\mathrm{Re}(z) < \frac{1}{2}$, move z left of $\mathrm{Re}(w) = \frac{1}{2}$, return contour to original. $1-z$ is right of $\mathrm{Re}(w) = \frac{1}{2}$, adding residue

$$\mathrm{Res}_{s=1-z} \frac{E_s \otimes E_{1-s}}{(s-z)^2 \, (s-(1-z))^2}$$

$$= \; \frac{-(E_{1-z} \otimes E_z)'}{(1-2z)^2} - 2\frac{E_{1-z} \otimes E_z}{(1-2z)^3}$$

Negative residue at $s = z$ and residue at $s = 1 - z$ do not *cancel*, but are *equal*. Summary: for $\mathrm{Re}(z) < \frac{1}{2}$

$$\mathfrak{P}_z(g,h) \;=\; \frac{1/\langle 1,1\rangle}{\lambda_z^2} \;+\; \sum_F \frac{F(g)\,\overline{F}(h)}{(\lambda_F - \lambda_z)^2}$$

$$+\;\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\,E_{1-s}(h)\;ds}{(\lambda_s - \lambda_z)^2}$$

$$-\;2\,\frac{(E_z \otimes E_{1-z})'}{(2z-1)^2} + 4\,\frac{E_z \otimes E_{1-z}}{(2z-1)^3}$$

Poles of E_z are at $z = \frac{\rho}{2}$ for non-trivial zeros ρ of $\zeta_k(s)$.

Note: meromorphic continuation of the $L^2(\Gamma\backslash G/K)$ function $\mathfrak{P}_z$ is immediately *not* L^2 to the left of $\mathrm{Re}(z) = \frac{1}{2}$. Not in any global automorphic Sobolev space.

E_s not in any global automorphic Sobolev space.

Perron integrals Recall: for $\sigma > 0$ and $\theta \gg 1$,

$$\frac{\theta^\ell \ell!}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{2zT}\, dz}{z(z+\theta)(z+2\theta)\ldots(z+\ell\theta)}$$

$$= \begin{cases} (1 - e^{-2\theta T})^\ell & (\text{for } T > 0) \\ 0 & (\text{for } T < 0) \end{cases}$$

Fundamental solution

$$\frac{r\, e^{-(2z-1)r}}{(2z-1)\sinh r} = \frac{r\, e^r}{(2z-1)\sinh r} \cdot e^{-2z \cdot r}$$

has $(2z-1)$ in denominator, insert compensating factor in numerator of Perron: let

$$R(z) = R_{\ell,\theta}(z) = \frac{(2z-1)}{z(z+\theta)(z+2\theta)\ldots(z+\ell\theta)}$$

Modified Perron applied to sum for $\mathfrak{P}_z(g,h)$

$$\frac{\theta^\ell\,\ell!}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathfrak{P}_z(g,h)\,e^{2zT}\,R(z)\,dz$$

$$= \sum_{\gamma\in\Gamma\,:\,r=d(g,\gamma h)<T}\frac{r\,e^r}{\sinh r}\cdot(1-e^{-2\theta(T-r)})^\ell$$

for $\sigma \gg 1$, ℓ large. Right side is *smoothed, weighted counting*, still a $\Gamma\times\Gamma$-invariant continuous function on $G/K \times G/K$.

Increasing ℓ improves convergence of all the relevant integrals.

Increasing θ makes the counting better-and-better approximate the discrete cut-off.

In terms of counting, there is conflict between θ and ℓ, since increasing θ improves counting, and increasing ℓ degrades it.

Perron integrals on spectral terms *Idea:* for suitable *meromorphic* function f in left half-plane,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(z)\cdot e^{2zT}\,R(z)\,dz$$

$$= \sum_{\text{poles } w \text{ of } f,\, \mathrm{Re}\,(w)<\sigma} \mathrm{Res}_{z=w} f(z)\, e^{2zT}\,R(z)$$

$$+ \sum_{j=0}^{\ell} \mathrm{Res}_{z=-j\theta}\; f(z)\, e^{2zT}\,R(z)$$

Residual term produces

$$e^{2T}\cdot\Big[T\cdot\frac{2R(1)}{\langle 1,1\rangle} \;+\; \frac{\partial}{\partial z}\Big[\frac{R(z)/\langle 1,1\rangle}{z^2}\Big]\Big|_{z=1}\Big]$$

$$+\,\mathrm{Res}_{z=0}\frac{e^{2zT}\,R(z)/\langle 1,1\rangle}{(z-1)^2\cdot z^2}$$

$$+\sum_{j=1}^{\ell} e^{-2j\theta T}\cdot\frac{\mathrm{Res}_{z=-j\theta}R(z)}{(-j\theta-1)^2\cdot(-j\theta)^2}$$

Cuspidal term produces similar

$$F(g)\,\overline{F}(h) \;\cdot\; \Big[\, T\, e^{2s_F T} \cdot \frac{2R(s_F)}{(2s_F-1)^2}$$

$$+\, T\, e^{2(1-s_F)T} \cdot \frac{2R(1-s_F)}{(1-2s_F)^2}$$

$$+\, e^{2s_F T} \cdot \frac{\partial}{\partial z}\Big[\frac{R(z)}{(z-(1-s_F))^2}\Big]\Big|_{z=s_F}$$

$$+\, e^{2(1-s_F)T} \cdot \frac{\partial}{\partial z}\Big[\frac{R(z)}{(z-s_F)^2}\Big]\Big|_{z=1-s_F}$$

$$+\, \frac{R(0)}{(-s_F)^2 \cdot (-(1-s_F))^2}$$

$$+\, \sum_{j=1}^{\ell} e^{-2j\theta T} \cdot \frac{\mathrm{Res}_{z=-j\theta} R(z)}{(-j\theta - s_F)^2 \cdot (-j\theta - (1-s_F))^2}\Big]$$

Contribution of continuous spectrum is more complicated.

For both $\mathrm{Re}(z) > \frac{1}{2}$ and $\mathrm{Re}(z) < \frac{1}{2}$ literal integral

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_s(g)\, E_{1-s}(h)\, ds}{(\lambda_s - \lambda_z)^2}$$

is holomorphic. **However**, meromorphic continuation of integral to $\mathrm{Re}(z) < \frac{1}{2}$ introduces

$$2\, \frac{(E_z \otimes E_{1-z})'}{(2z-1)^2} + 4\, \frac{E_z \otimes E_{1-z}}{(2z-1)^3} \quad (\text{for } \mathrm{Re}(z) < \tfrac{1}{2})$$

exhibiting the poles in the latter half-plane. Poles of $R(z)\, e^{2zT}$ at $0, -\theta, -2\theta, \ldots, -\ell\theta$ are simple, corresponding residues

$$\sum_{j=0}^{\ell} e^{-2j\theta T} \cdot \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_s(g)\, E_{1-s}(h)$$

$$\times \; \frac{\mathrm{Res}_{z=-j\theta} R(z)}{(-j\theta - s)^2 \cdot (-j\theta - (1-s))^2}\, ds$$

Extra terms in meromorphic continuation of continuous spectrum are holomorphic at $0, -\theta, -2\theta, \ldots, -\ell\theta$, so residues

$$\sum_{j=0}^{\ell} e^{-2j\theta T} \cdot \Big[\Big(-\, 2\, \frac{(E_z \otimes E_{1-z})'|_{z=-j\theta}}{(2(-j\theta)-1)^2}$$

$$+\, 4\, \frac{E_{-j\theta} \otimes E_{1+j\theta}}{(2(-j\theta)-1)^3}\Big) \cdot \mathrm{Res}_{z=-j\theta}\Big(\frac{R(z)}{(z-1)^2 \cdot z^2}\Big)\Big]$$

More interestingly, non-trivial zeros ρ of $\zeta_k(s)$ produce poles of E_z at $z = \rho/2$, in $0 < \mathrm{Re}(z) < \frac{1}{2}$, producing residues

$$\sum_{\rho} \mathrm{Res}_{z=\frac{\rho}{2}}\Big(\Big[-\, 2\, \frac{(E_z \otimes E_{1-z})'}{(2z-1)^2}$$

$$+4\, \frac{E_z \otimes E_{1-z}}{(2z-1)^3}\Big] \cdot e^{2zT}\, R(z)\Big)$$

For zero ρ order $m(\rho)$, resulting function of T of form $P_\rho(T) \cdot e^{\rho T}$ with $P_\rho(T)$ polynomial of degree $m(\rho)$.

Comments:

It is easier to justify *finite* contour shifting to $\mathrm{Re}(z) = \sigma > 1$ or to $\mathrm{Re}(z) = \frac{1}{2} + \delta$, by Hadamard three-circles. But this does not tell *composition* of smaller terms.

Perron integral gives *full asymptotics*, but requires **threading** horizontal contours between poles of *extra terms*.

That is, need N and $t_n \to +\infty$ such that

$$\sup_{0 \le \sigma \le 1} \frac{1}{|\zeta_k(\sigma + it_n)|} \ll_N |t_n|^N$$

Convexity bound, Hadamard factorization, and smidgen of cleverness suffice.

RH gives a better exponent, but is overkill.

Smoothed, weighted lattice-point counting
Thus, there are constructive rational functions A, B such that, for *fixed* g, h in G/K,

$$\sum_{\gamma\in\Gamma\,:\,r=d(g,\gamma h)<T} \frac{r\,e^r}{\sinh r}\cdot(1-e^{-2\theta(T-r)})^{\ell}$$

$$= \quad [A(1)T+B(1)]\,\frac{e^{2T}}{\langle 1,1\rangle}$$

$$+\;\sum_F [A(s_F)T+B(s_F)]\,e^{2s_F T}$$

$$+\;\sum_F [A(1-s_F)T+B(1-s_F)]\,e^{2(1-s_F)T}$$

$$+\quad\sum_{\text{non}-\text{trivial zero }\rho} P_\rho(T)\,e^{\rho T}$$

$$+\quad\text{(explicit, bounded in } T\text{)}$$

where P_ρ is a polynomial of degree $m(\rho)$.

What did we not do?

What worries did we not have?

We did not try to estimate sup norms of cuspforms, nor of Eisenstein series. Admittedly, pointwise estimation of Eisenstein series for $SL_2(\mathbb{Z})$ is worse than for $SL_2(\mathbb{Z}[i])$: the former's values on $\mathrm{Re}(s) = \frac{1}{2}$ are L-function values on the critical line, while the latter's values on $\mathrm{Re}(s) = \frac{1}{2}$ are products of L-function values on the *edge* of the critical strip.

That is, the convexity bound for $SL_2(\mathbb{Z}[i])$ Eisenstein series on $\mathrm{Re}(s) = \frac{1}{2}$ is *correct*, while for $SL_2(\mathbb{Z})$ subconvexity result already prove it inaccurate, with or without Lindelöf.

We did not worry about *where* sups occur, as function of eigenvalue.

We did not convert the weighted, smoothed counting into a *classical* unweighted, not-smoothed counting. Rather, we gave an exact formula for the counting as it stands, showing exactly how all the spectral terms enter.

Comment/update: At Newark, in addition to the computation narrated here, I also reported briefly on related unfinished computations which seemed to have a provocative outcome.

Completion of those computations, perhaps not surprisingly, but of-course-disappointingly, new information on $GL(1)$ phenomena is *not* obtained by coercing $GL(2)$ phenomena in this particular fashion.

That is, the obvious, and perhaps *crude*, suppression of $GL(2)$ cuspidal contributions in a $GL(2)$ identity (circuitously) leads back to literal $GL(1)$ computations, in a form that does not refer to any automorphic phenomena on $GL(2)$.

In fancier terms: suppressing some volatility in a trace formula leads to a boring outcome.

A little disappointing, but also enlightening.