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Non-existence of tensor products of Hilbert spaces

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Tensor products of infinite-dimensional Hilbert spaces do not exist.

That is, for infinite-dimensional Hilbert spaces V, W, **there is no** Hilbert space X and continuous bilinear map $j : V \times W \longrightarrow X$ such that, for every continuous bilinear $V \times W \longrightarrow Y$ to a Hilbert space Y, there is a unique continuous linear $X \longrightarrow Y$ fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ V \times W \arrow[u, "j"] \arrow[r] & Y \end{tikzcd}$$

That is, *there is no tensor product in the category of Hilbert spaces and continuous linear maps.*

Yes, it *is* possible to put an inner product on the *algebraic* tensor product $V \otimes_{\mathrm{alg}} W$, by

$$\langle v \otimes w,\, v' \otimes w' \rangle \;=\; \langle v, v' \rangle \cdot \langle w, w' \rangle$$

and extending. The *completion* H of $V \otimes_{\mathrm{alg}} W$ with respect to the associated norm, often denoted $V \widehat{\otimes} W$, is a Hilbert space, identifiable with *Hilbert-Schmidt* operators $V \longrightarrow W^*$. *However*, this Hilbert space fails to have the universal property in the categorical characterization of tensor product, as we see below. This Hilbert space H is important in its own right, but is widely incorrectly portrayed as a tensor product.

The *non-existence* of tensor products of infinite-dimensional Hilbert spaces is important *in practice*, not only as a cautionary tale[1] about naive category theory, insofar as it leads to Grothendieck's idea of *nuclear spaces*. A main feature of nuclear spaces is that they *do* admit tensor products. The original explicit example of this was Schwartz' *Kernel Theorem*, although earlier discussions of extending differential operators to subspaces of L^2 can be recast in such terms using Sobolev spaces.

Proof: First, we show that the Hilbert-Schmidt tensor product $H = V \widehat{\otimes} W$ cannot be a Hilbert-space tensor product. For simplicity, suppose that V, W are *separable*, in the sense of having countable Hilbert-space bases. [2]

Choice of such bases allows an identification of W with the continuous linear Hilbert space dual V^* of V. Then we have the continuous bilinear map $V \times V^* \longrightarrow \mathbb{C}$ by $v \times \lambda \longrightarrow \lambda(v)$. The *algebraic* tensor product $V \otimes_{\mathrm{alg}} W$ injects to $H = V \widehat{\otimes} V^*$, and the image is identifiable with the *finite-rank* maps $V \longrightarrow V$. The linear map $T : H \longrightarrow \mathbb{C}$ induced on the image of $V \otimes_{\mathrm{alg}} V^*$ is *trace*. If $H = V \widehat{\otimes} V^*$ were a Hilbert-space tensor product, the trace map would extend continuously to it from finite-rank operators. However, there are many Hilbert-Schmidt operators that are not of trace class. For example, letting e_i be an orthonormal basis, the element

$$\sum_n \frac{1}{n} \cdot e_n \otimes e_n \quad \in \quad V \widehat{\otimes} V^*$$

[1] Many of us are not accustomed to worry about *existence* of objects defined by universal mapping properties, because we learned set-theoretic *constructions* of them, thus proving their existence, long before becoming aware of mapping-property characterizations. Indeed, much as naive set theory does not lead to paradoxes without some effort, naive category theory's recharacterization of objects not far from prior experience rarely describes non-existent objects. Nevertheless, examples such as the present one are genuine.

[2] Large Hilbert spaces, with uncountable Hilbert space bases, are rare, in practice. In any case, the same argument succeeds in almost the same fashion for Hilbert spaces with Hilbert space bases of the same infinite cardinality. When the cardinalities are distinct, the Axiom of Choice enables comparison of the two cardinalities, and we can identify the dual of the smaller with a subspace of the larger, and then a similar argument applies.

does not have a finite trace, since $\sum_{n\le N} 1/n \sim \log N$. In other words, the difficulty is that

$$T\Big(\sum_{a\le n\le b} \frac{1}{n}\cdot e_n\otimes e_n\Big) \;=\; \sum_{a\le n\le b} \frac{1}{n}\cdot T(e_n\otimes e_n) \;=\; \sum_{a\le n\le b} \frac{1}{n}$$

Thus, the partial sums of $\sum_n \frac{1}{n}\, e_n\otimes e_n$ form a Cauchy sequence, but the values of T on the partial sums go to $+\infty$. Thus, the Hilbert-Schmidt tensor product cannot be a Hilbert-space tensor product.

Now we show that no *other* Hilbert space can be a tensor product, by comparing to the Hilbert-Schmidt tensor product.

Let $V\times W \longrightarrow X$ be a purported Hilbert-space tensor product, and, again, let W be the dual of V, without loss of generality. By assumption, the continuous bilinear injection $V\times V^* \longrightarrow V\widehat{\otimes}V^*$ induces a unique continuous linear map $T: X\longrightarrow H$ fitting into a commutative diagram

$$\begin{tikzcd}
X \arrow[ddrr, dashed, bend left, "T"] & & \\
& V\otimes_{\mathrm{alg}} W \arrow[ul, dashed] \arrow[dr, dashed] & \\
V\times W \arrow[uu] \arrow[ur] \arrow[rr] & & H
\end{tikzcd}$$

The linear map $V\otimes_{\mathrm{alg}}V^* \longrightarrow H$ is injective, since H is a completion of $V\otimes_{\mathrm{alg}}V^*$. Thus, unsurprisingly, $V\otimes_{\mathrm{alg}}V^* \longrightarrow X$ is necessarily injective. The uniqueness of the linear induced maps implies that the image of $V\otimes_{\mathrm{alg}}V^*$ is *dense* in X. Also, $T: X\longrightarrow H$ is the identity on the copies of $V\otimes_{\mathrm{alg}}V^*$ imbedded in X and H.

Let $T^*: H\longrightarrow X$ be the adjoint of T, defined by

$$\langle x, T^*y\rangle_X \;=\; \langle Tx, y\rangle_H$$

On the imbedded copies of $V\otimes_{\mathrm{alg}}V^*$

$$\langle v\otimes\lambda, T^*(w\otimes\mu)\rangle_X \;=\; \langle T(v\otimes\lambda), w\otimes\mu\rangle_H \;=\; \langle v\otimes\lambda, w\otimes\mu\rangle_H \qquad (\text{for } v,w\in V \text{ and } \lambda,\mu\in V^*)$$

Given $v\in V$ and $\lambda\in V^*$, the orthogonal complement $(v\otimes\lambda)^\perp$ is the closure of the span of monomials $v'\otimes\lambda'$ where *either* $v'\perp v$ *or* $\lambda'\perp\lambda$. For such $v'\otimes\lambda'$,

$$0 \;=\; \langle v'\otimes\lambda', v\otimes\lambda\rangle_H \;=\; \langle T(v'\otimes\lambda'), v\otimes\lambda\rangle_H \;=\; \langle v'\otimes\lambda', T^*(v\otimes\lambda)\rangle_X$$

Thus, for any monomial $v\otimes\lambda$, the image $T^*(v\otimes\lambda)$ is a scalar multiple of $v\otimes\lambda$. The same is true of monomials $(v+w)\otimes(\lambda+\mu)$. Taking v,w linearly independent and λ,μ linearly independent and expanding shows that the scalars do not depend on v,λ. Thus, T^* is a scalar on $V\otimes_{\mathrm{alg}}V^*$.

That is, there is a (necessarily real) constant C such that

$$C\cdot\langle v\otimes\lambda, w\otimes\mu\rangle_X \;=\; \langle v\otimes\lambda, T^*(w\otimes\mu)\rangle_X \;=\; \langle T(v\otimes\lambda), w\otimes\mu\rangle_H \;=\; \langle v\otimes\lambda, w\otimes\mu\rangle_H$$

since T identifies the imbedded copies of $V\otimes_{\mathrm{alg}}V^*$. That is, up to the constant C, the inner products from X and H restrict to the same hermitian form on $V\otimes_{\mathrm{alg}}V^*$. Thus, any putative X differs from H only by scaling. However, we saw that the natural pairing $V\times V^*\longrightarrow\mathbb{C}$ does not factor through a continuous linear map $H\longrightarrow\mathbb{C}$, because there exist Hilbert-Schmidt maps not of trace class.

Thus, there is no tensor product of infinite-dimensional Hilbert spaces. ///
