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Standard p-adic integrals for $GL(2)$

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The p-adic integrals evaluated here were explicitly introduced only in the later 20th century, starting with Tate and Iwasawa in 1950, by MacDonald in the 1950s, in the 1960's by Gelfand and Piatetski-Shapiro, Jacquet, Shalika, and then by Jacquet-Langlands in 1970, although shadows of them appeared long ago in the work of Lagrange, Legendre, Gauss, Galois, and Dirichlet. From our vantage, they are analogues of classical archimedean integrals.

Throughout, we compute intertwinings under the tacit assumption that parameters are in a range such that integrals are absolutely convergent. Amusingly, the parameter values for unitary principal series are at the *edge* of the region of absolute convergence, so a further analytic continuation argument is needed.

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1. *Normalizations of L-functions*

Some details of the larger context are helpful in appreciating the p-adic computations. This also gives a more detailed preview of some of the later computations.

The classical description of the L-function attached to a holomorphic modular form

$$f_0(z) \;=\; \sum_{n\geq 1} a_n\, e^{2\pi i n z}$$

of level 1 and of weight $\kappa \in 2\mathbb{Z}$ on the upper half-plane is

$$\Lambda(s, f_0) \;=\; \int_0^\infty y^s\; f_0(iy)\,\frac{dy}{y} \;=\; (2\pi)^{-s}\,\Gamma(s)\,\sum_{n\geq 1}\frac{a_n}{n^s}$$

The functional equation $f_0(-1/z) = z^\kappa \cdot f_0(z)$ of f_0 gives the corresponding functional equation

$$\Lambda(\kappa - s, f_0) \;=\; \Lambda(s, f_0)$$

For various reasons, a normalization that gives a functional equation $s \longleftrightarrow 1-s$ is more convenient. This is *almost* accomplished by thinking in terms of the associated automorphic form f on the Lie group, in this case given by

$$f(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}) \;=\; y^{\kappa/2}\cdot f_0(x+iy)$$

If we were to take the Mellin transform of this, the functional equation would be with respect to $s \longleftrightarrow -s$, which would be better, in that it would depend less upon the specific local data. And, in principle, the normalization of coordinates cannot matter. However, patterns of more sophisticated phenomena in both local representation theory and in a global theory of automorphic forms agitate for a correct uniform notion of *critical strip* for the L-function. It turns out that a good modern normalization is

$$\Lambda(s, f) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\; f(iy)\,\frac{dy}{y} \;=\; \int_0^\infty y^{s-\frac{1}{2}+\frac{\kappa}{2}}\; f_0(iy)\,\frac{dy}{y} \;=\; (2\pi)^{-(s-\frac{1}{2}+\frac{\kappa}{2})}\;\Gamma(s-\tfrac{1}{2}+\tfrac{\kappa}{2})\sum_{n\geq 1}\frac{a_n}{n^{s-\frac{1}{2}+\frac{\kappa}{2}}}$$

$$= \; (2\pi)^{-\frac{\kappa-1}{2}} \; \cdot \; (2\pi)^{-s}\,\Gamma(s-\tfrac{1}{2}+\tfrac{\kappa}{2}) \sum_{n\geq 1} \frac{a_n/n^{\frac{\kappa-1}{2}}}{n^s}$$

Generally, the *standard* L-function[1] attached to a cuspform f on GL_2 over a number field k, including the gamma factor, is the Mellin transform

$$\Lambda(s,f) \;=\; \int_{\mathbb{J}/k^\times} |y|^{s-\frac{1}{2}}\, f\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} d^\times y \;=\; \int_{\mathbb{J}} |y|^{s-\frac{1}{2}}\, W_f\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} d^\times y$$

where W_f is the global Whittaker function for f, namely,

$$W_f(g) \;=\; \int_{k\backslash\mathbb{A}} \overline{\psi}(x)\; f(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g)\, dx$$

with fixed non-trivial character ψ. In this normalization, the L-function has a functional equation under $s \longleftrightarrow 1-s$. Uniqueness of local Whittaker models implies that W_f factors over primes $W_f = \bigotimes_v W_v$. Thus, letting π_v denote the (irreducible) representation of $GL_2(k_v)$ generated by f, the v^{th} Euler factor of $\Lambda(s,f)$ is given by the local Mellin transform

$$L_v(s,\pi_v) \;=\; \int_{k_v^\times} |y|^{s-\frac{1}{2}}\; W_v\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} d^\times y$$

For example, for $k_v \approx \mathbb{R}$, for a holomorphic discrete series representation π_v of weight $\kappa \in 2\mathbb{Z}$, the Whittaker function for the lowest K_v-type is

$$W_v\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} \;=\; y^{\kappa/2}\, e^{-2\pi y} \qquad\qquad (\text{for } y>0)$$

Thus, the local L-function (gamma factor) in this normalization is

$$L_v(s,\pi_v) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\; y^{\kappa/2}\, e^{-2\pi y}\,\frac{dy}{y} \;=\; \int_0^\infty y^{s+\frac{\kappa-1}{2}}\; e^{-2\pi y}\,\frac{dy}{y} \;=\; (2\pi)^{-(s+\frac{\kappa-1}{2})}\;\Gamma(s+\frac{\kappa-1}{2})$$

At spherical finite places v, the local Whittaker function is given below by the easiest case of the Shintani-Kato-Casselman-Shalika formula,

$$W_v(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}) \;=\; \begin{cases} \psi(x)\cdot \dfrac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} & (\text{for } n=\operatorname{ord} y \geq 0) \\ 0 & (\text{for } n = \operatorname{ord} y < 0)\end{cases}$$

where $\alpha\beta = \omega(\varpi)/q$, with q the residue field cardinality, ω is the central character, ϖ a local parameter, and ψ is the fixed additive character specifying the Whittaker model. Thus, at good finite primes,

$$L_v(s,\pi_v) \;=\; \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty} q^{-n(s-\frac{1}{2})}(\alpha^{n+1}-\beta^{n+1})$$

$$=\; \frac{1}{\alpha-\beta}\cdot\Big(\frac{\alpha}{1-\alpha q^{-(s-\frac{1}{2})}} - \frac{\beta}{1-\beta q^{-(s-\frac{1}{2})}}\Big) \;=\; \frac{1}{\big(1-\alpha q^{-(s-\frac{1}{2})}\big)\big(1-\beta q^{-(s-\frac{1}{2})}\big)}$$

[1] This integral is most properly termed a *zeta integral*, rather than *L-function*, since only an optimal choice of cuspform within an irreducible gives good local factors, especially at bad primes. The discussion of finite bad primes is not the point here.

$$= \frac{1}{1 - \sqrt{q}(\alpha+\beta)\, q^{-s} + \omega(\varpi) q^{-2s}}$$

We can write the local L-factor in the form

$$L_v(s, \pi_v) \;=\; \frac{1}{\big(1 - A\, q^{-s}\big)\big(1 - B\, q^{-s}\big)} \;=\; \frac{1}{\det\big(1 \;-\; q^{-s} \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\big)}$$

with

$$A \;=\; q^{\frac{1}{2}}\alpha \qquad\qquad B \;=\; q^{\frac{1}{2}}\beta \qquad\qquad \text{(up to permutations)}$$

In this notation, the local factor of the **symmetric square** L-function at a spherical finite prime is

$$\frac{1}{\det\big(1 \;-\; q^{-s}\cdot \mathrm{Sym}^2 \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\big)} \;=\; \frac{1}{\det\big(1 \;-\; q^{-s}\cdot \begin{pmatrix} A^2 & 0 & 0 \\ 0 & AB & 0 \\ 0 & 0 & B^2 \end{pmatrix}\big)}$$

$$= \frac{1}{(1 - A^2\, q^{-s})\,(1 - AB\, q^{-s})\,(1 - B^2\, q^{-s})}$$

$$= \frac{1}{(1 - q\alpha^2\, q^{-s})\,(1 - q\alpha\beta\, q^{-s})\,(1 - q\beta^2\, q^{-s})} \;=\; \frac{1}{(1 - q\alpha^2\, q^{-s})\,(1 - \omega(\varpi) q^{-s})\,(1 - q\beta^2\, q^{-s})}$$

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2. *Unramified principal series*

Let k be a finite extension of some non-archimedean completion of $\mathbb{Q}$, over which it is assumed *unramified*, for convenience. [2] Let $G = GL_2(k)$. For applications to automorphic forms, the most important irreducible representations of G are the *spherical* representations, meaning irreducibles possessing non-zero vectors fixed under the compact subgroup [3] $K = GL_2(\mathfrak{o})$ of G. It is non-trivial [4] that every spherical smooth representation is a *subrepresentation* of one of the *smooth* [5] *unramified principal series* representations, described just below, and is also a *quotient* of one such. [6]

[2] The assumption that a non-archimedean local field k is absolutely unramified over $\mathbb{Q}_p$ is convenient because then the local trace pairing $k \times k \to \mathbb{Q}_p$ by $\alpha \times \beta \to \mathrm{tr}\,^k_{\mathbb{Q}_p}(\alpha\beta)$ makes the local integers be their own dual module. Let ψ_o be the additive character on $\mathbb{Q}_p$ extended by the homomorphism property and continuity from $\psi_o(p^{-\ell}) = e^{-2\pi i p^{-\ell}}$. Let ψ be the additive character on k obtained by composing ψ_o with trace. This ψ has the convenient feature that the indicator function of the local integers is mapped to itself by Fourier transform.

[3] In fact, $K = GL_2(\mathfrak{o})$ is *maximal* compact in G. We do not use this fact. Further, every maximal compact subgroup in G is *conjugate* to K. We do not use this fact, either. For $G = GL_2(k)$, proofs of these assertions are not difficult, but are not high priority.

[4] Arguments special to GL_2 can be made to show that every spherical representation imbeds in an unramified principal series, but the best general argument is the *Borel-Matsumoto theorem*, whose proof uses non-trivial facts about affine *buildings* attached to p-adic reductive groups.

[5] As usual, a representation of a totally disconnected group G such as $GL_2(k)$ on a complex vector space V is *smooth* if the isotropy subgroup of every $v \in V$ is *open* in G. This turns out to be the correct analogue for p-adic groups of differentiability conditions for Lie groups.

[6] The two unramified principal series representations of which π is a subrepresentation and a quotient are certainly isomorphic in the typical case that these unramified principal series are irreducible.

A character $\alpha : k^\times \to \mathbb{C}^\times$ is **unramified** when it is trivial on the local units $\mathfrak{o}^times$. In that case, α factors through $k^\times/\mathfrak{o}^\times$, so is a function of the norm, and can be written as [7]

$$\alpha(y) \;=\; |y|^s \qquad\qquad \text{(for complex } s\text{)}$$

The most straightforward model for the naively normalized **unramified principal series** of representations of G induced from the subgroup P of upper-triangular matrices, with trivial central character, is [8]

$$I_\chi^{\rm nf} \;=\; \{f \in C^\infty(G) \text{ with } f(p\cdot g) = \chi(p)\, f(g) \text{ for all } p \in P \text{ and } g \in G\} \qquad \left(\text{where } \chi\begin{pmatrix} a & * \\ 0 & d\end{pmatrix} = |a/d|^s\right)$$

As usual, the smoothness condition $f \in C^\infty(G)$ on a totally disconnected G is really a *local constancy* condition that, given $g \in G$, there is an open subgroup U of G such that

$$f(g\cdot u) \;=\; f(g) \qquad\qquad \text{(for all } u \in U\text{)}$$

It is easy to consider unramified principal series with not-necessarily-trivial central character ω, namely

$$I_\chi^{\rm nf} \;=\; \{f \in C^\infty(G) : f(p\cdot g) = \chi(p)\, f(g)\} \qquad \left(\text{where } \chi\begin{pmatrix} a & * \\ 0 & d\end{pmatrix} = \omega(d)\cdot|a/d|^s\right)$$

Most of the computations below are qualitatively the same for non-trivial central character as for trivial, with a slight increase in notational burden.

Another non-trivial fact [9] is that typically $I_\chi^{\rm nf}$ is *irreducible*. Thus, the spherical representations of p-adic GL_2 are essentially unramified principal series. The advantage in shifting our attention from spherical representations to unramified principal series is that the latter are very explicitly described in a form convenient for computations and applications.

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3. *The simplest integral: intertwining among principal series*

The simplest integral related to non-archimedean $G = GL_2(k)$ computes the effect of a natural, frequently appearing, intertwining operator among unramified principal series. Namely, when it converges, the integral

$$T_s f(g) \;=\; \int_N f(wng)\, dn \qquad\qquad \left(\text{where } w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},\ N = \begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix}\right)$$

gives a G-hom

$$T_s \;:\; I_s^{\rm nf} \longrightarrow I_{1-s}^{\rm nf}$$

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[7] The complex parameter s is ambiguous by integer multiples of $2\pi i/q$, where $q > 1$ is a generator for the (discrete) group of non-zero values of the norm on k.

[8] Although we need an explicit model for these representations in order to carry out some standard computations, the best definition of $I_\chi^{\rm nf}$ is as an object that makes a suitable form of *Frobenius Reciprocity* hold, namely, for every *smooth* representation π of G,

$$\mathrm{Hom}_G(\pi, I_\chi^{\rm nf}) \;\approx_{\mathbb{C}}\; \mathrm{Hom}_P(\mathrm{Res}_P^G\pi, \chi)$$

where P is the subgroup of upper-triangular matrices. That is, $I_\chi^{\rm nf}$ is the image of χ under the functor *adjoint* to the forgetful functor Res_P^G that converts G-representations to P-representations. Verification that the present construction succeeds in exhibiting such an adjoint is not difficult, but not our point.

[9] A precise statement about typical irreducibility of unramified principal series representations is best considered a corollary of the Borel-Matsumoto theorem, as in [Casselman 1980].

The shift in index from s to $1-s$ is verified by changing variables: replace n by mnm^{-1}.

The specific integral we will compute is related to the image $T_s\varphi_s^o$ of the *spherical* vector $\varphi_s^o \in I_s^{\rm nf}$, that is, the right $K = GL_2(\mathfrak{o})$-invariant function in $I_s^{\rm nf}$ taking value 1 at $1 \in G$. Certainly a G-hom maps spherical vectors to spherical vectors. By the Iwasawa decomposition, there is a unique such function in $I_s^{\rm nf}$.

Thus, $T_s\varphi_s^o$ is a constant multiple of φ_{1-s}^o. This constant is

$$T_s\varphi_s^o(1) \;=\; \int_N \varphi_s^o(wn)\,dn$$

where w is the longest Weyl element. To evaluate the integral, use the p-adic Iwasawa decomposition, and observe the two regimes: first, for $n \in N \cap K$, the product wn is already inside K, so $\varphi_s^o(wn) = 1$, by its definition. For the opposite case that $n \notin K$, take $x \notin \mathfrak{o}$, and note that

$$w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \;=\; w\begin{pmatrix} 0 & x \\ -x^{-1} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix} = \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} -x^{-2} & * \\ 0 & 1 \end{pmatrix} \qquad (\text{as } \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \in K)$$

Thus, in this case,

$$\varphi_s^o(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \;=\; \chi\begin{pmatrix} -x^{-1} & * \\ 0 & -x \end{pmatrix} \;=\; |-x^{-2}|^s \;=\; |x|^{-2s} \qquad (\text{for } x \notin \mathfrak{o})$$

Together,

$$\int_N \varphi_s^o(wn)\,dn \;=\; \text{meas}\,(\mathfrak{o})\cdot 1 + \int_{x\notin\mathfrak{o}} |x|^{-2s}\,dx \;=\; 1 + \sum_{n\ge 1} \text{meas}\,(\mathfrak{o}^\times \cdot \varpi^{-n})\cdot |\varpi^{-n}|^{-2s}$$

$$=\; 1 + \sum_{n\ge 1} \frac{q-1}{q} q^n \cdot (q^n)^{-2s} \;=\; 1 + \frac{q-1}{q}\sum_{n\ge 1}(q^{1-2s})^n \;=\; 1 + 1 - \frac{1}{q}\frac{q^{1-2s}}{1-q^{1-2s}}$$

$$=\; \frac{1 - q^{1-2s} + (1-\frac{1}{q})q^{1-2s}}{1-q^{1-2s}} \;=\; \frac{1-q^{-2s}}{1-q^{1-2s}} \;=\; \frac{\zeta_v(2s-1)}{\zeta_v(2s)}$$

Thus, with the Levi-component part included to show how the parameter of the unramified principal series changes, we have

$$\int_N f(wn\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix})\,dn \;=\; |y|^{1-2s}\cdot\frac{1-q^{-2s}}{1-q^{1-2s}}$$

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4. *Spherical Whittaker functions (trivial central character)*

This classical computation is the very simplest instance of the Kato-Shintani-Casselman-Shalika formula for spherical Whittaker convolutions on GL_n and on reductive groups generally.

The Whittaker function attached to a spherical representation modeled by an unramified principal series $I_\chi^{\rm nf}$ is the image under a natural intertwining operator [10] from $I_\chi^{\rm nf}$ to the Whittaker space attached to a fixed character ψ on N, namely

$$W_\chi^{\rm nf}(g) \;=\; \int_N \overline{\psi}(n)\,\varphi_\chi^{\rm sph}(w\cdot n\cdot g)\,dn \qquad (\text{where } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$

[10] The intertwining operator from a principal series to the Whittaker space is essentially unique (up to scalars). This is visible from Mackey-Bruhat double-coset considerations, namely, that $P\backslash G/N$ has just two elements, and one of them cannot support a non-trivial intertwining operator.

where $\varphi_\chi^{\rm sph}$ is the spherical vector in $I_{\chi^{\rm nf}}$ normalized by $\varphi_\chi^{\rm sph}(1) = 1$. Since this intertwining operator yields $W_\chi^{\rm nf}$ left N-equivariant and right K-invariant, it suffices to evaluate $W_\chi^{\rm nf}$ on M. In fact, since the central character is determined by that of the principal series, to know $W_\chi^{\rm nf}$ it suffices to evaluate

$$W_\chi^{\rm nf}\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} \;=\; \int_k \overline{\psi}(x)\,\varphi_\chi^{\rm sph}(w\cdot n_x\cdot m_y)\,dx \qquad \text{(where } n_x = \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \text{ and } m_y = \begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix}\text{)}$$

Roughly: unless y is integral, local cancellation due to ψ will cause the integrand to vanish entirely. For y integral, there is still a local cancellation effect for $\mathrm{ord}\,x$ large negative. At the edge of this regime, some cancellation will occur without annihilating the integrand entirely. Thus, the integral will be equal to a finite geometric series with slightly different beginning and ending terms.

For k absolutely unramified, the *standard* character $\psi : k \to \mathbb{C}^\times$ is trivial on the local integers $\mathfrak{o}$ but non-trivial on $\varpi^{-1}\mathfrak{o}$, with ϖ a local parameter. [11] We make this assumption on ψ. An unramified character χ with trivial central character is of the form

$$\chi\begin{pmatrix} a & * \\ 0 & d\end{pmatrix} \;=\; |a/d|^s \qquad \text{(with } s\in\mathbb{C}\text{)}$$

First,

$$w\cdot n_x\cdot m_y \;=\; w\cdot m_y\cdot n_{x/y} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & y\end{pmatrix}\cdot w\cdot\begin{pmatrix} 1 & x/y \\ 0 & 1\end{pmatrix}$$

Thus, with trivial central character,

$$\varphi(w\cdot n_x\cdot m_y) \;=\; \chi\begin{pmatrix} 1 & 0 \\ 0 & y\end{pmatrix}\cdot\varphi(w\cdot n_{x/y})$$

Thus,

$$W(m_y) \;=\; \chi\begin{pmatrix} 1 & 0 \\ 0 & y\end{pmatrix}\cdot\int_k \overline{\psi}(x)\,\varphi(w\cdot n_{x/y})\,dx \;=\; |y|^{1-s}\cdot\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx$$

by replacing x by xy, producing a change-of-measure constant of $|y|$.

We compute the latter integral. For $y\notin\mathfrak{o}$, the character

$$x \to \overline{\psi}(xy)$$

is non-trivial on $\mathfrak{o}$. On the other hand, $n_t\in K$ for $t\in\mathfrak{o}$, and φ is right K-invariant, so

$$\varphi(w\cdot n_x\cdot n_t) \;=\; \varphi(w\cdot n_x) \qquad \text{(for } t\in\mathfrak{o}\text{)}$$

Thus, we have a standard vanishing argument by change of variables, as follows.

$$\int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x)\,dx \;=\; \int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x{\cdot}n_t)\,dx \;=\; \int_k \overline{\psi}((x{-}t)y)\,\varphi(w{\cdot}n_x)\,dx \;=\; \psi(ty)\int_k \overline{\psi}(xy)\,\varphi(w{\cdot}n_x)\,dx$$

by replacing x by $x-t$. Since $y\notin\mathfrak{o}$, there is $t\in\mathfrak{o}$ such that $\psi(ty)\neq 1$. Thus,

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; 0 \qquad \text{(for } y\notin\mathfrak{o}\text{)}$$

[11] Even when k is absolutely ramified, sometimes there is a character ψ trivial on $\mathfrak{o}$ and non-trivial on $\varpi^{-1}\mathfrak{o}$. This issues is not the point here.

Now take $y \in \mathfrak{o}$. Here we need to compute $\varphi(w \cdot n_x)$ via the p-adic Iwasawa decomposition of wn_x: right modulo K,

$$w \cdot n_x \;=\; \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \;=\; \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \;=\; \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} & (\text{for ord}\, x \le 0) \\ \;=\; 1 & (\text{for ord}\, x \ge 0) \end{cases}$$

Thus, under the convention[12] that the measure of $\mathfrak{o}$ is 1, break the integral over $k - \mathfrak{o}$ into $\mathfrak{o}^\times$ orbits:

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy) \cdot 1\, dx \;+\; \int_{k-\mathfrak{o}} \overline{\psi}(xy)\, |x^{-2}|^s\, dx$$

For fixed $y \in \mathfrak{o}$, for $\text{ord}\, xy < -1$, the map

$$x \to x \cdot (1 + \varpi u) \qquad\qquad (\text{with } u \in \mathfrak{o})$$

leaves $\varphi(wn_x)$ invariant, but

$$\overline{\psi}(x(1+\varpi u)y) \;=\; \overline{\psi}(xy) \cdot \overline{\psi}(xy\varpi \cdot u)$$

Since $xy\varpi \notin \mathfrak{o}$, the character

$$u \to \overline{\psi}(xy\varpi \cdot u) \qquad\qquad (\text{for } u \in \mathfrak{o})$$

is non-trivial, so the integral in x over such an $(1 + \varpi\mathfrak{o})$-orbit must vanish. Thus,

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy) \cdot 1\, dx + \int_{0 > \text{ord}x \ge -1-\text{ord}\, y} \overline{\psi}(xy)\, |x|^{-2s}\, dx$$

Further, there is no cancellation due to ψ except when $\text{ord}\, xy = -1$, so

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} 1\, dx + \int_{-\text{ord}\, y \le \text{ord}x < 0} |x|^{-2s}\, dx + \int_{\text{ord}x = -(1+\text{ord}\, y)} \overline{\psi}(xy)\, |x|^{-2s}\, dx$$

Let $n = \text{ord}\, y$ and q the residue field cardinality. In the last integral, $|x|^{-2s}$ is constant, and

$$\int_{\text{ord}x = -(1+\text{ord}\, y)} \overline{\psi}(xy)\, dx \;=\; \int_{\text{ord}x \ge -(1+\text{ord}\, y)} \overline{\psi}(xy)\, dx - \int_{\text{ord}x \ge -\text{ord}\, y} \overline{\psi}(xy)\, dx \;=\; 0 - \text{meas}\,(y^{-1}\mathfrak{o}) \;=\; -q^n$$

since the first integral is the integral of a non-trivial character. Thus,

$$\int_{\text{ord}x = -(1+\text{ord}\, y)} \overline{\psi}(xy)\, |x|^{-2s}\, dx \;=\; -q^n \cdot (q^{(1+n)})^{-2s}$$

Thus, so far,

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; 1 + \frac{q-1}{q} \sum_{\ell=1}^{n} q^\ell \cdot (q^\ell)^{-2s} \;-\; q^n \cdot (q^{(1+n)})^{-2s}$$

using the comparison of additive and multiplicative measures

$$\text{meas}\,(\varpi^{-\ell}\mathfrak{o}^\times) \;=\; q^\ell \cdot \frac{q-1}{q}$$

[12] The choice of additive Haar measure on k giving $\mathfrak{o}$ total measure 1 is compatible with other reasonable conventions when k is absolutely unramified, but not otherwise.

Summing the finite geometric series,

$$1 + \frac{q-1}{q} \cdot \frac{q^{1-2s} - (q^{n+1})^{1-2s}}{1-q^{1-2s}} \;-\; q^n \cdot q^{-(n+1)2s}$$

To see how this should simplify, let $X = q^{1-2s}$. Then the whole is

$$1 + \frac{q-1}{q} \cdot \frac{X - X^{n+1}}{1-X} \;-\; \frac{X^{n+1}}{q}$$

$$= \frac{q(1-X) + (q-1)(X - X^{n+1}) - (1-X)X^{n+1}}{q(1-X)}$$

$$= \frac{q - qX + qX - X - qX^{n+1} + X^{n+1} - X^{n+1} + X^{n+2}}{q(1-X)} = \frac{q - X - qX^{n+1} + X^{n+2}}{q(1-X)}$$

$$= \frac{1 - \frac{1}{q}X - X^{n+1} + \frac{1}{q}X^{n+2}}{1-X} = \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X}$$

Also, express $|y|^{1-s}$ in terms of X,

$$|y|^{1-s} \;=\; (q^{-n})^{1-s} \;=\; (q^{-\frac{n}{2}})^{2-2s} \;=\; q^{-\frac{n}{2}} \cdot (q^{-\frac{n}{2}})^{1-2s} \;=\; q^{-\frac{n}{2}} \cdot X^{-\frac{n}{2}}$$

Thus,

$$W^{\mathrm{nf}}(m_y) \;=\; |y|^{1-s} \cdot \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X} \;=\; q^{-\frac{n}{2}} \cdot X^{-\frac{n}{2}} \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X}$$

$$= (1-\tfrac{1}{q}X) \cdot q^{-\frac{n}{2}} \cdot \frac{X^{-\frac{n+1}{2}} - X^{\frac{n+1}{2}}}{X^{-\frac{1}{2}} - X^{\frac{1}{2}}} \;=\; (1-\tfrac{1}{q}X) \cdot \frac{(1/qX)^{\frac{n+1}{2}} - (X/q)^{\frac{n+1}{2}}}{(1/qX)^{\frac{1}{2}} - (X/q)^{\frac{1}{2}}}$$

Let

$$\alpha = (1/qX)^{\frac{1}{2}} = q^{\frac{1}{2}(-(1+1-2s))} = q^{-1+s} \qquad\qquad \beta = (X/q)^{\frac{1}{2}} = q^{\frac{1}{2}(1-2s-1)} = q^{-s}$$

Note that

$$\alpha \cdot \beta \;=\; 1/q$$

Then

$$W_s^{\mathrm{nf}}(m_y) \;=\; (1-q^{-2s}) \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

Normalizing this Whittaker function to take value 1 at the identity encourages us to discard the leading constant $1-q^{-2s}$. Further, the unramified principal series representations satisfy a relation under $s \to 1-s$ in the naive normalization, suggesting replacing s by $\frac{1}{2} + i\mu$. Thus, with a less naive normalization,

$$W_{i\mu}(m_y) \;=\; \frac{1}{1-q^{-2(\frac{1}{2}+i\mu)}} \cdot W^{\mathrm{nf}}_{\frac{1}{2}+i\mu}(m_y) \;=\; \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \qquad\qquad (\text{where } \alpha = q^{-\frac{1}{2}+i\mu} \text{ and } \beta = q^{-\frac{1}{2}-i\mu})$$

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5. *Spherical Whittaker functions (non-trivial central character)*

Now we re-do the computation of the Whittaker function for an unramified principal series, allowing arbitrary unramified central character. This adds some further notational clutter, but no new ideas. The discussion will repeat the previous one, but be less verbose.

The Whittaker function attached to an unramified principal series I_χ^{nf} is the image under a natural intertwining operator from I_χ^{nf} to the Whittaker space attached to a fixed character ψ on N, namely

$$W_\chi^{\mathrm{nf}}(g) \;=\; \int_N \overline{\psi}(n)\, \varphi_\chi^{\mathrm{sph}}(w\cdot n\cdot g)\, dn \qquad\qquad (\text{where } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$

where $\varphi_\chi^{\mathrm{sph}}$ is the spherical vector in $I_{\chi^{\mathrm{nf}}}$ normalized by $\varphi_\chi^{\mathrm{sph}}(1)=1$. Since this intertwining operator yields W_χ^{nf} left N-equivariant and right K-invariant, and since the central character is determined by that of the principal series, to know W_χ^{nf} it suffices to evaluate

$$W_\chi^{\mathrm{nf}}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;=\; \int_k \overline{\psi}(x)\, \varphi_\chi^{\mathrm{sph}}(w\cdot n_x\cdot m_y)\, dx \qquad\qquad (\text{where } n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } m_y = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\ )$$

The *standard* character $\psi : k \to \mathbb{C}^\times$ is trivial on the local integers $\mathfrak{o}$ but non-trivial on $\varpi^{-1}\mathfrak{o}$, with ϖ a local parameter. An unramified character χ with central character ω is of the form

$$\chi\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \;=\; \omega(d)\cdot |a/d|^s \qquad\qquad (\text{with } s\in\mathbb{C})$$

First,

$$w\cdot n_x\cdot m_y \;=\; w\cdot m_y\cdot n_{x/y} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\cdot w\cdot \begin{pmatrix} 1 & x/y \\ 0 & 1 \end{pmatrix}$$

Thus,

$$\varphi(w\cdot n_x\cdot m_y) \;=\; \chi\begin{pmatrix} 1 & \\ & y \end{pmatrix}\cdot\varphi(w\cdot n_{x/y}) \;=\; \omega\begin{pmatrix} y & \\ & y \end{pmatrix}\chi\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix}\cdot\varphi(w\cdot n_{x/y})$$

For convenience, write

$$\omega(y) \;=\; \omega\begin{pmatrix} y & \\ & y \end{pmatrix}$$

Thus,

$$W(m_y) \;=\; \omega(y)\,\chi\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}\cdot\int_k \overline{\psi}(x)\,\varphi(w\cdot n_{x/y})\,dx \;=\; \omega(y)\,|y|^{1-s}\cdot\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx$$

by replacing x by xy. For $y\notin\mathfrak{o}$, the character

$$x\to\overline{\psi}(xy)$$

is non-trivial on $\mathfrak{o}$. On the other hand, $n_t\in K$ for $t\in\mathfrak{o}$, and φ is right K-invariant, so

$$\varphi(w\cdot n_x\cdot n_t) \;=\; \varphi(w\cdot n_x) \qquad\qquad (\text{for } t\in\mathfrak{o})$$

Thus, we have a standard vanishing argument by change of variables, as follows.

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; \int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x\cdot n_t)\,dx \;=\; \int_k \overline{\psi}((x-t)y)\,\varphi(w\cdot n_x)\,dx \;=\; \psi(ty)\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx$$

by replacing x by $x-t$. Since $y\notin\mathfrak{o}$, there is $t\in\mathfrak{o}$ such that $\psi(ty)\neq 1$. Thus,

$$\int_k \overline{\psi}(xy)\,\varphi(w\cdot n_x)\,dx \;=\; 0 \qquad\qquad (\text{for } y\notin\mathfrak{o})$$

Now take $y \in \mathfrak{o}$. Here we need to compute $\varphi(w \cdot n_x)$ via the p-adic Iwasawa decomposition of wn_x: right modulo K,

$$w \cdot n_x \;=\; \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \;=\; \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x^{-1} & 1 \end{pmatrix} \;=\; \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} & (\text{for } \operatorname{ord} x \leq 0) \\ \;=\; 1 & (\text{for } \operatorname{ord} x \geq 0) \end{cases}$$

Thus, under the convention that the measure of $\mathfrak{o}$ is 1, break the integral over $k - \mathfrak{o}$ into $\mathfrak{o}^\times$ orbits:

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy) \cdot 1\, dx \;+\; \int_{k-\mathfrak{o}} \overline{\psi}(xy)\, \omega(x) |x^{-2}|^s\, dx$$

For fixed $y \in \mathfrak{o}$, for $\operatorname{ord} xy < -1$, the map

$$x \to x \cdot (1 + \varpi u) \qquad\qquad (\text{with } u \in \mathfrak{o})$$

leaves $\varphi(wn_x)$ invariant, but

$$\overline{\psi}(x(1+\varpi u)y) \;=\; \overline{\psi}(xy) \cdot \overline{\psi}(xy\varpi \cdot u)$$

Since $xy\varpi \notin \mathfrak{o}$, the character

$$u \to \overline{\psi}(xy\varpi \cdot u) \qquad\qquad (\text{for } u \in \mathfrak{o})$$

is non-trivial, so the integral in x over such an $(1 + \varpi\mathfrak{o})$-orbit must vanish. Thus,

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} \overline{\psi}(xy) \cdot 1\, dx + \int_{0 > \operatorname{ord} x \geq -1 - \operatorname{ord} y} \overline{\psi}(xy)\, \omega(x) |x|^{-2s}\, dx$$

Further, there is no cancellation due to ψ except when $\operatorname{ord} xy = -1$, so

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; \int_{\mathfrak{o}} 1\, dx + \int_{-\operatorname{ord} y \leq \operatorname{ord} x < 0} \omega(x) |x|^{-2s}\, dx + \int_{\operatorname{ord} x = -(1 + \operatorname{ord} y)} \overline{\psi}(xy)\, \omega(x) |x|^{-2s}\, dx$$

Let $n = \operatorname{ord} y$ and q the residue field cardinality. In the last integral, $\omega(x)|x|^{-2s}$ is constant, and

$$\int_{\operatorname{ord} x = -(1 + \operatorname{ord} y)} \overline{\psi}(xy)\, dx \;=\; \int_{\operatorname{ord} x \geq -(1 + \operatorname{ord} y)} \overline{\psi}(xy)\, dx - \int_{\operatorname{ord} x \geq -\operatorname{ord} y} \overline{\psi}(xy)\, dx \;=\; 0 - \operatorname{meas}(y^{-1}\mathfrak{o}) \;=\; -q^n$$

since the first integral is the integral of a non-trivial character. Thus,

$$\int_{\operatorname{ord} x = -(1 + \operatorname{ord} y)} \overline{\psi}(xy)\, \omega(x) |x|^{-2s}\, dx \;=\; -q^n \cdot \omega(\varpi^{-(n+1)}) (q^{(1+n)})^{-2s}$$

Thus, so far,

$$\int_k \overline{\psi}(xy)\, \varphi(w \cdot n_x)\, dx \;=\; 1 + \frac{q-1}{q} \sum_{\ell=1}^{n} q^\ell \cdot \omega(\varpi^{-\ell}) (q^\ell)^{-2s} \;-\; q^n \cdot \omega(\varpi^{-(n+1)}) (q^{(1+n)})^{-2s}$$

using the comparison of additive and multiplicative measures

$$\operatorname{meas}(\varpi^{-\ell}\mathfrak{o}^\times) \;=\; q^\ell \cdot \frac{q-1}{q}$$

Summing the finite geometric series,

$$1 + \frac{q-1}{q} \cdot \frac{\omega(\varpi^{-1}) q^{1-2s} - (\omega(\varpi^{-(n+1)}) q^{n+1})^{1-2s}}{1 - \omega(\varpi^{-1}) q^{1-2s}} \;-\; q^n \cdot \omega(\varpi^{-(n+1)}) q^{-(n+1)2s}$$

To see how this should simplify, let $X = \omega(\varpi^{-1})q^{1-2s}$. Then the whole is

$$1 + \frac{q-1}{q} \cdot \frac{X - X^{n+1}}{1-X} \; - \; \frac{X^{n+1}}{q}$$

$$= \; \frac{q(1-X) + (q-1)(X - X^{n+1}) - (1-X)X^{n+1}}{q(1-X)}$$

$$= \; \frac{q - qX + qX - X - qX^{n+1} + X^{n+1} - X^{n+1} + X^{n+2}}{q(1-X)} \; = \; \frac{q - X - qX^{n+1} + X^{n+2}}{q(1-X)}$$

$$= \; \frac{1 - \frac{1}{q}X - X^{n+1} + \frac{1}{q}X^{n+2}}{1-X} \; = \; \frac{(1-\frac{1}{q}X)\,(1-X^{n+1})}{1-X}$$

$$= \; (1 - \frac{\omega(\varpi^{-1})q^{1-2s}}{q}) \cdot \frac{1 - \omega(\varpi^{-1})^{n+1}\,(q^{1-2s})^{n+1}}{1 - \omega(\varpi^{-1})\,q^{1-2s}} \; = \; (1 - \omega(\varpi^{-1})q^{-2s}) \cdot \frac{1 - \omega(\varpi^{-1})^{n+1}\,(q^{1-2s})^{n+1}}{1 - \omega(\varpi^{-1})\,q^{1-2s}}$$

Put back the leading factor $\omega(y)|y|^{1-s} = \omega(\varpi^n)(q^{-n})^{1-s}$ to obtain

$$W^{\rm nf}(m_y) \; = \; \omega(\varpi^n)(q^{-n})^{1-s}(1 - \omega(\varpi^{-1})q^{-2s}) \cdot \frac{1 - \omega(\varpi^{-1})^{n+1}\,(q^{1-2s})^{n+1}}{1 - \omega(\varpi^{-1})\,q^{1-2s}}$$

$$= \; (1 - \omega(\varpi^{-1})q^{-2s}) \cdot \frac{\big(\omega(\varpi)q^{-(1-s)}\big)^{n+1} - \big(q^{-s}\big)^{n+1}}{\omega(\varpi)q^{-(1-s)} - q^{-s}}$$

Let

$$\alpha = \omega(\varpi)q^{-(1-s)} \qquad\qquad \beta = q^{-s}$$

Note that

$$\alpha \cdot \beta \; = \; \frac{\omega(\varpi)}{q} \qquad\qquad \text{(with central character } \omega)$$

Then

$$W^{\rm nf}(m_y) \; = \; (1 - \omega(\varpi^{-1})q^{-2s}) \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

Normalizing this Whittaker function to take value 1 at the identity encourages us to discard the leading constant $1 - \omega(\varpi^{-1})q^{-2s}$. Further, the unramified principal series representations satisfy a relation under $s \to 1-s$ in the naive normalization, suggesting replacing s by $\frac{1}{2} + i\mu$. Thus, with a less naive normalization,

$$W_\chi(m_y) = \frac{1}{1 - \omega(\varpi^{-1})q^{-2(\frac{1}{2}+i\mu)}} \cdot W^{\rm nf}_\chi(m_y) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \qquad\qquad (\text{with } \alpha = \omega(\varpi)q^{-\frac{1}{2}+i\mu},\; \beta = q^{-\frac{1}{2}-i\mu})$$

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6. *Spherical Mellin transforms*

The v-adic local factor in the Mellin transform representation of standard L-functions for GL_2 is

$$\int_{k^\times} |y|^{s-\frac{1}{2}}\, W \begin{pmatrix} y & \\ & 1 \end{pmatrix} \, dy \qquad\qquad (s \in \mathbb{C}, \text{ measure is multiplicative Haar})$$

where W is the non-naively normalized Whittaker function $W = W_\chi$ just computed. The shift by $\frac{1}{2}$ in the exponent is as in the discussion of normalization of L-functions. Since the integrand is $\mathfrak{o}^\times$-invariant, the integral over $k^\times$ is

$$\int_{k^\times} \; = \; \int_{k^\times/\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \; = \; \int_{k^\times/\mathfrak{o}^\times} \qquad\qquad (\text{giving } \mathfrak{o}^\times \text{ measure 1})$$

with $k^\times/\mathfrak{o}^\times$ having counting measure. Thus, the local Mellin transform is

$$\int_{k^\times} |y|^{s-\frac{1}{2}}\, W\begin{pmatrix} y & \\ & 1\end{pmatrix}\, dy \;=\; \sum_{n=0}^{\infty} |\varpi^n|^{s-\frac{1}{2}}\, \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

$$= \sum_{n=0}^{\infty} (q^{-n})^{s-\frac{1}{2}}\, \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \qquad \text{(where } \alpha=\omega(\varpi)q^{-\frac{1}{2}+i\mu},\ \beta=q^{-\frac{1}{2}-i\mu}\text{)}$$

Summing the geometric series gives

$$\frac{1}{\alpha-\beta}\cdot\Big(\frac{\alpha}{1-\alpha q^{-(s-\frac{1}{2})}} - \frac{\beta}{1-\beta q^{-(s-\frac{1}{2})}}\Big) \;=\; \frac{1}{\alpha-\beta}\cdot\frac{\alpha-\alpha\beta q^{-(s-\frac{1}{2})}-\beta+\alpha\beta q^{-(s-\frac{1}{2})}}{(1-\alpha q^{-(s-\frac{1}{2})})\,(1-\beta q^{-(s-\frac{1}{2})})}$$

$$= \frac{1}{(1-\alpha q^{-(s-\frac{1}{2})})\,(1-\beta q^{-(s-\frac{1}{2})})} \;=\; \frac{1}{1-\sqrt{q}(\alpha+\beta)\,q^{-s}+q\alpha\beta\,q^{-2s}} \;=\; \frac{1}{1-\sqrt{q}(\alpha+\beta)\,q^{-s}+\omega(\varpi)\,q^{-2s}}$$

since, as in the computation of Whittaker functions, $\alpha\beta=\omega(\varpi)/q$, with central character ω.

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7. *Spherical Rankin-Selberg integrals*

The local integral appearing in a Rankin-Selberg convolution is

$$\int_{Z\backslash G} \varepsilon(g)\, W_1(g)\, W_2(g)\, dg$$

where ε_s is a vector in a principal series, and the W_j's are Whittaker functions. Locally at absolutely unramified finite places where ε is the normalized spherical vector in the s^{th} unramified principal series, where both W_j's are spherical, all with trivial central characters, the integral over $Z\backslash G$ is an integral over $Z\backslash G/K$ where K is maximal compact. Then the integral is over $Z\backslash P$ where P has *left* Haar measure,

$$\int_{k^\times} |y|^{s-1}\, W_1\begin{pmatrix} y & \\ & 1\end{pmatrix}\, W_2\begin{pmatrix} y & \\ & 1\end{pmatrix}\, dy \;=\; \sum_{n\geq 0} |\varpi^\ell|^{s-1}\, \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\cdot\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}$$

where $\alpha,\beta,\gamma,\delta$ are complex numbers with

$$\alpha\beta \;=\; \gamma\delta \;=\; 1/q \qquad \text{(for trivial central characters)}$$

With $X=|\varpi|^{s-1}=q^{-(s-1)}$, this is

$$\sum_{n\geq 0} X^n\, \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\cdot\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} \;=\; \frac{1}{(\alpha-\beta)(\gamma-\delta)}\,\Big(\frac{\alpha\gamma}{1-\alpha\gamma X}-\frac{\alpha\delta}{1-\alpha\delta X}-\frac{\beta\gamma}{1-\beta\gamma X}+\frac{\beta\delta}{1-\beta\delta X}\Big)$$

$$= \frac{1}{(\alpha-\beta)(\gamma-\delta)}\,\Big(\frac{\alpha\gamma-\alpha^2\gamma\delta X-\alpha\delta+\alpha^2\gamma\delta X}{(1-\alpha\gamma X)\,(1-\alpha\delta X)} - \frac{\beta\gamma-\beta^2\gamma\delta X-\beta\delta+\beta^2\gamma\delta X}{(1-\beta\gamma X)\,(1-\beta\delta X)}\Big)$$

$$= \frac{1}{(\alpha-\beta)}\,\Big(\frac{\alpha}{(1-\alpha\gamma X)\,(1-\alpha\delta X)} - \frac{\beta}{(1-\beta\gamma X)\,(1-\beta\delta X)}\Big)$$

$$= \frac{1}{(\alpha-\beta)}\cdot\frac{\Big(\alpha-\frac{\gamma}{q}X-\frac{\delta}{q}X+\frac{\beta}{q^2}\,X^2\Big)-\Big(\beta-\frac{\gamma}{q}X-\frac{\delta}{q}X+\frac{\alpha}{q^2}\,X^2\Big)}{(1-\alpha\gamma X)\,(1-\alpha\delta X)\,(1-\beta\gamma X)\,(1-\beta\delta X)}$$

$$= \frac{1}{(\alpha-\beta)}\cdot\frac{(\alpha-\beta)-(\alpha-\beta)\frac{1}{q^2}\,X^2}{(1-\alpha\gamma X)\,(1-\alpha\delta X)\,(1-\beta\gamma X)\,(1-\beta\delta X)}$$

$$= \frac{1 - \frac{1}{q^2} X^2}{(1-\alpha\gamma X)\,(1-\alpha\delta X)\,(1-\beta\gamma X)\,(1-\beta\delta X)}$$

$$= \frac{1 - q^{-2s}}{(1-\alpha\gamma q^{-(s-1)})\,(1-\alpha\delta q^{-(s-1)})\,(1-\beta\gamma q^{-(s-1)})\,(1-\beta\delta q^{-(s-1)})}$$

Since

$$\alpha\beta \;=\; \gamma\delta \;=\; \frac{1}{q}$$

the shift in the exponent of q can reasonably be absorbed:

$$(1-q^{-2s}) \cdot \frac{1}{\det\Big(1_4 \;-\; q^{-s}\cdot \begin{pmatrix} \alpha\sqrt{q} & \\ & \beta\sqrt{q} \end{pmatrix} \otimes \begin{pmatrix} \gamma\sqrt{q} & \\ & \delta\sqrt{q} \end{pmatrix}\Big)}$$

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8. *Spherical symmetric square via Rankin-Selberg*

In the non-archimedean local Rankin-Selberg integral, when the two Whittaker functions are the same, that is, when[13]

$$\gamma = \alpha \text{ and } \delta = \beta \qquad \text{or} \qquad \gamma = \beta \text{ and } \delta = \alpha$$

there is simplification: the integral of the last section becomes

$$\frac{1 - q^{-2s}}{(1-\alpha^2 q^{-(s-1)})\,(1-\frac{1}{q} q^{-(s-1)})\,(1-\beta^2 q^{-(s-1)})\,(1-\frac{1}{q} q^{-(s-1)})}$$

$$= \frac{1 - q^{-2s}}{(1-\alpha^2 q^{-(s-1)})\,(1-q^{-s})\,(1-\beta^2 q^{-(s-1)})\,(1-q^{-s})}$$

$$= \frac{1-q^{-2s}}{1-q^{-s}} \cdot \frac{1}{(1-\alpha^2 q^{-(s-1)})\,(1-q^{-s})\,(1-\beta^2 q^{-(s-1)})} \;=\; \frac{1-q^{-2s}}{1-q^{-s}} \cdot \frac{1}{\det\Big(1_3 \;-\; q^{-s}\cdot \begin{pmatrix} \alpha^2 q & & \\ & 1 & \\ & & \beta^2 q \end{pmatrix}\Big)}$$

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[13] This could be put more elegantly, as the requirement that the conjugacy class of $\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ is the same as the conjugacy class of $\begin{pmatrix} \gamma & \\ & \delta \end{pmatrix}$.