(*October 16, 2009*)

Characterization of differential operators

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

Differential operators obviously do not increase support when applied to test functions. The converse is certainly not clear. [Peetre 1959,60] proved this, incorporating corrections from L. Carleson. We follow [Helgason 1984] pp 236-238, who adapts the argument from [Narasimhan 1968].

[0.0.1] Theorem: Let V be a smooth manifold. A not-necessarily-continuous linear map $D : C_c^\infty(V) \to C_c^\infty(V)$ that *does not increase supports* is a differential operator with smooth coefficients.

Proof: First, claim that the non-increase of support property implies that, for a test function f and a point x, for any test function φ identically 1 on a neighborhood of x, suitable truncation does not affect D, in the sense that

$$(Df)(x) \;=\; \big(D(\varphi f)\big)(x)$$

Indeed, $f = \varphi f + (1-\varphi)f$, and D is linear, so

$$Df \;=\; D(\varphi f) + D((1-\varphi)f)$$

The non-increase of support implies that $D((1-\varphi)f)(x) = 0$, yielding the claim.

This truncation property immediately allows us to consider the corresponding local problem, of operators on open subsets of Euclidean spaces, without loss of generality.

Next, the non-increase of support allows an extension of D to *all* smooth functions on V by using cut-off functions: given smooth f and a point x, let φ be a test function identically 1 on a neighborhood of x, and define $Df(x) = D(\varphi f)(x)$. The latter is well-defined by the previous claim.

Let $|f|_{U,m}$ be the sup on U of sups of the derivatives of f of orders $\le m$.

Next, claim that for f smooth on U with derivatives of order $\le m$ vanishing at 0, for every $\varepsilon > 0$ there is a smooth function g vanishing identically in a neighborhood of 0, coinciding exactly with f *outside* a larger neighborhood of 0, such that $|f-g|_{U,m} < \varepsilon$. Let φ be a smooth function identically 0 on $|x| \le \frac{1}{2}$, identically 1 for $|x| \ge 1$, and $0 \le \varphi \le 1$ everywhere. Then consider the family of modifications of f given by

$$g_\delta(x) \;=\; \varphi(x/\delta)\cdot f(x) \qquad\qquad \text{(for } \delta > 0 \text{ small)}$$

Each g_δ agrees with f outside the δ-ball B_δ at 0. It would suffice to prove

$$\lim_{\delta\to 0} |f - g_\delta|_{B_\delta,m} \;=\; 0$$

Since f vanishes to order m at 0,

$$\lim_{\delta\to 0} |f|_{B_\delta,m} \;=\; 0$$

so we must prove that

$$\lim_{\delta\to 0} |g_\delta|_{B_\delta,m} \;=\; 0$$

For multi-index α, apply Leibniz' rule to the α^{th} derivative of g_δ:

$$g_\delta^{(\alpha)}(x) \;=\; \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \delta^{-|\alpha|}\varphi^{(\beta)}(x/\delta)\, f^{(\gamma)}(x)$$

Thus,

$$|g_\delta^{(\alpha)}(x)| \;\ll\; \sum_{\beta+\gamma=\alpha} \delta^{-|\beta|}\,|f^{(\gamma)}(x)| \qquad\qquad \text{(with } x \in B_\delta)$$

with implied constant independent of f and δ. The derivative $f^{(\gamma)}$ vanishes to order $m - |\gamma|$ at 0, so, from the Taylor expansion of f at 0,

$$\sup_{B_\delta} |f^{(\gamma)}| \;=\; o(\delta^{m-|\gamma|})$$

Thus,

$$\sup_{B_\delta} |g_\delta^{(\alpha)}(x)| \;=\; o\Big(\sum_{\beta+\gamma=\alpha} \delta^{m-|\beta|-|\gamma|}\Big) \;=\; o(\delta^{m-|\alpha|})$$

Thus, as claimed, $|f - g_\delta|_{B_\delta, m} \to 0$.

Next, claim a somewhat weaker *continuity* assertion than the theorem, namely, that for every point x_o there is a sufficiently small neighborhood U of x_o, integer m, such that

$$|Df|_{U,0} \;\ll\; |f|_{U,m} \qquad\qquad (\text{for } f \in C_c^\infty(U - \{x_o\}))$$

with the implied constant independent of f. This follows by a diagonal argument: if this failed at some x_o, then for given compact-closure neighborhood U_0 of x_o there is $f_1 \in C_c^\infty(U_o - \{x_o\})$ such that

$$|Df_1|_0 \;\geq\; 2^2 \cdot |f_1|_1$$

Let U_1 be the zero-set of f_1, so $U_0 - \overline{U}_1$ is a neighborhood of x_o, and there is $f_2 \in C_c^\infty(U_0 - \overline{U}_1 - \{x_o\})$ such that

$$|Df_1|_0 \;\geq\; 2^4 \cdot |f_2|_2$$

By induction, obtain open sets U_i with $\overline{U}_i \cap \overline{U}_j = \phi$ for $i, j \geq 1$, and test functions

$$f_i \;\in\; C_c^\infty\big(U_0 - \overline{U}_1 - \ldots - \overline{U}_{i-1} - \{x_o\}\big)$$

with

$$|Df_i|_0 \;\geq\; 2^{2i} \cdot |f_i|_i$$

Then the sum

$$\sum_i \frac{f_i}{2^i \cdot |f|_i}$$

converges and gives a test function, equal to the i^{th} summand $f_i/(2^i \cdot |f|_i)$ on U_i. The linearity and non-increase of support of D imply that

$$Df\Big|_{U_i} \;=\; \frac{1}{2^i \cdot |f|_i} \cdot Df_i\Big|_{U_i}$$

Thus, there exists $x_i \in U_i$ such that $Df(x_i) > 2^i$. But f is continuous and compactly supported, so this is impossible, proving the claim.

Next, thinking in terms of that last weak continuity, we prove a *local* result: for a neighborhood U of a point x, under the continuity hypothesis

$$|Df|_{U,0} \;\ll\; |f|_{U,m}$$

on a sufficiently small neighborhood of x, D is a differential operator with smooth coefficients. For the proof of this, for each $x \in U$ and multi-index α, let

$$P_{x,\alpha}(y) \;=\; (x-y)^\alpha \;=\; (x_1 - y_1)^{\alpha_1} \ldots (x_n - y_n)^{\alpha_n}$$

For $f \in C_c^\infty(U)$ and fixed $x \in U$, consider a subsum of the Taylor expansion of f near x,

$$F \;=\; f - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} f^{(\alpha)}(x) \cdot P_{\alpha,x}$$

This F vanishes to order m at x. As shown above, given $\varepsilon > 0$ there is a test function Φ_ε vanishing identically in a neighborhood of x (depending upon ε), agreeing identically with F outside a larger neighborhood of x (depending on ε), and with $|F - \Phi_\varepsilon|_m \le \varepsilon$. The continuity assumption gives $|D(F - \Phi_\varepsilon)|_0 \to 0$ as $\varepsilon \to 0$. The non-increase of support implies that each $D\Phi_\varepsilon$ vanishes identically near x. Thus, $|DF(x)| < \varepsilon$ for every $\varepsilon > 0$, so $DF(x) = 0$. Thus, for each $x \in U$,

$$Df(x) \;=\; \sum_{|\alpha|\le m} \frac{1}{\alpha!} f^{(\alpha)}(x) \cdot DP_{\alpha,x}(x)$$

To understand $b_\alpha(x) = DP_{\alpha,x}(x)$, observe that it is a sum of terms $P_\beta(x)\, y^\beta$ with P_β a polynomial. By linearity of D,

$$D(\sum_\beta P_\beta(x) \cdot y^\beta) \;=\; \sum_\beta P_\beta(x) \cdot D(y^\beta)$$

By hypothesis $D(y^\beta)$ is a test function, so the diagonal

$$DP_{x,\alpha}(x) \;=\; \sum_\beta P_\beta(x) \cdot D(x^\beta)$$

is a finite sum of polynomial multiples of test functions, and is a test function itself. Thus, the expression for $Df(x)$ exhibits it as a differential operator with smooth coefficients on U.

Finally, we reduce the general question of expressibility of D to the local one, essentially by a partition of unity argument. At each $x \in V$, let U_x be a small-enough neighborhood of x, m_x an integer, so that we have a continuity bound

$$|Df|_{U_x,0} \;\ll\; |f|_{U_x,m_x} \qquad (\text{for } f \in C_c^\infty(U_x - \{x\}))$$

with implied constant independent of f. For an open $U \subset V$ with compact closure $\overline{U} \subset V$, take a finite subcover $U_{x_1}, \ldots, U_{x_n}$ of the opens U_x. Let $\{\varphi_j\}$ be a partition of unity subordinate to the cover $U_{x_1}, \ldots, U_{x_n}$ and $V - \overline{U}$ of V. For f a test function on the set

$$U' \;=\; U - \{x_1\} - \ldots - \{x_n\}$$

certainly

$$f \;=\; \sum_{j=1}^{n+1} \varphi_j \cdot f \;=\; \sum_{j=1}^{n} \varphi_j \cdot f$$

and each $\varphi_j f$ satisfies a corresponding continuity bound. Expanding the derivatives of $\varphi_j f$ by Leibniz, we find that f itself satisfies such a continuity bound on U_{x_j}, and, therefore, satisfies a uniform continuity bound throughout U'. Thus, on U', D is a differential operator with smooth coefficients

$$Df(x) \;=\; \sum_j a_j(x) \cdot \Big(\frac{\partial}{\partial x}\Big)^\alpha f(x) \qquad (\text{for } x \in U',\, f \in C_c^\infty(U'))$$

In fact, the non-increase of support property allows us to extend the validity of this to $f \in C_c^\infty(U)$, at least for $x \in U'$: take $\varphi \in C_c^\infty(U')$ identically 1 near x and identically 0 near every x_i. Then $\varphi f \in C_c^\infty(U')$, and the property $D(\varphi f)(x) = Df(x)$ observed earlier gives

$$Df(x) \;=\; \sum_j a_j(x) \cdot \Big(\frac{\partial}{\partial x}\Big)^\alpha f(x) \qquad (\text{for } x \in U',\, f \in C_c^\infty(U))$$

Finally, because both sides of the last equation are continuous in x, this equality holds not merely for $x \in U'$, but for $x \in U$. This holds for every $\overline{U} \subset V$, so is valid on V. ///

[Helgason 1984] S. Helgason, *Groups and geometric analysis*, Academic Press, 1984.

[Narasimhan 1968] R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland, Amsterdam, 1968.

[Peetre 1959,1960] J. Peetre, *Une caractérization abstraite des opérateurs différentials*, Math. Scand. **7** (1959), 211-218; *Rectification*, *ibid* **8** (1960), 116-120.