(*February 19, 2005*)

Bernstein's Rationality Lemma

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Here we give a proof of a basic algebraic form of Bernstein's Rationality Lemma.

Let k be a field of characteristic zero, and V a k-vectorspace. A **linear system** (over k, with coefficients in V) is a set Ξ of ordered pairs (v_i, c_i) where $v_i \in V$ and $c_i \in k$ and $i \in I$ for some index set I. A **solution** to the linear system is $\lambda \in V^*$ so that, for all indices i,

$$\lambda(v_i) = c_i$$

where $V^* = \mathrm{Hom}_k(V, k)$ is the k-linear dual space of V. Obviously the set of all solutions is an affine subspace of V^*, so has a sense of **dimension**. A system Ξ is **homogeneous** if all the constants c_i are 0. In that case, the collection of solutions is a vector subspace of V^*. Two systems are **equivalent** if they have the same set of solutions.

Proposition: *(Existence, Uniqueness, and Rationality)* Let $\Xi = \{(v_i, c_i) : i \in I\}$ be a k-linear system with coefficients v_i in a k-vectorspace V. Suppose that there is at most one index $i_o \in I$ so that $c_{i_o} \neq 0$.
• If v_{i_o} does not lie in the k-span of $\{v_i : i \neq i_o\}$ then there is at least one solution to the linear system.
• If the coefficient vectors $\{v_i : i \in I\}$ span V, then there is at most one solution.

Proof: If v_{i_o} does not lie in the span of the other coefficient vectors, then (via the Axiom of Choice) there is a linear functional λ in the dual space V^* so that $\lambda(v_i) = 0$ for $i \neq i_o$, but $\lambda(v_o) = 1$. Then $c_{i_o}\lambda$ is a solution of the system Ξ.

Next, for two solutions λ, λ' of Ξ the difference $\mu = \lambda - \lambda'$ is a solution of the homogeneous system $\Xi_o = \{(v_i, 0)\}$. If the v_i span V, then the condition

$$\mu(v_i) = 0 \qquad \text{for all } i \in I$$

implies that $\mu = 0$. This is the uniqueness. ///

Let $\mathcal{O}$ be a commutative k-algebra, and suppose further that $\mathcal{O}$ is an integral domain. Let V be an $\mathcal{O}$-module. A **parametrized linear system** over $\mathcal{O}$ (or over X) with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$ is a collection Ξ of ordered pairs (μ_i, f_i) with $\mu_i \in M \otimes_{\mathcal{O}} \mathcal{M}$, $f_i \in \mathcal{M}$.

A **generic solution** to such a parametrized system is

$$\lambda \in (V \otimes_{\mathcal{O}} \mathcal{M})^* = \mathrm{Hom}_{\mathcal{M}}(V \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{M})$$

so that for all indices i

$$\lambda(\mu_i) = f_i$$

That is, a generic solution is simply a *solution* (in the previous sense) to the $\mathcal{M}$-linear system on the $\mathcal{M}$-vectorspace $V \otimes_{\mathcal{O}} \mathcal{M}$.

Note that we do not require the coefficient vectors μ_i to be in the module V, but only in $V \otimes_{\mathcal{O}} \mathcal{M}$, and likewise the f_i need not be in $\mathcal{O}$, but only in $\mathcal{M}$. Of course, the same collection of generic solutions would be obtained if each (μ_i, f_i) were replaced by $(g_i\mu_i, g_i f_i)$ for non-zero $g_i \in \mathcal{O}$. Thus, one *could* assume without loss of generality that all the μ_i are in V and the f_i are in $\mathcal{O}$, but it is not necessary to do so.

For $x \in X$, $f \in \mathcal{M}$ is **holomorphic at** x if f is in the local ring $\mathcal{O}_x$ of $\mathcal{O}$ at x. An element μ of $V \otimes_{\mathcal{O}} \mathcal{M}$ is **holomorphic at** x if $\mu \in M \otimes_{\mathcal{O}} \mathcal{O}_x$. A parametrized system $\Xi = \{(\mu_i, f_i)\}$ is **holomorphic at** x if for all indices i both f_i and μ_i are holomorphic at x. (From the definitions, *every* parametrized system is holomorphic at the generic point $0 \in X$).

Let k_x be the residue field $k_x = \mathcal{O}/x\mathcal{O}_x$ of $\mathcal{O}$ at x. For any $x \in X$ at which the parametrized system Ξ is holomorphic, we have the **associated pointwise system** Ξ_x, obtained by replacing all the elements

$\mu_i \in M \otimes_{\mathcal{O}} \mathcal{O}_x$ by their images in $V \otimes_{\mathcal{O}} k_x$, and likewise by replacing the $f_i \in \mathcal{O}_x$ by their images in k_x. Thus, Ξ_x is a k_x-linear system with coefficients in $V \otimes_{\mathcal{O}} k_x$.

A **pointwise solution** λ_x **at** x to the parametrized linear system Ξ is just a solution to the linear system Ξ_x. Thus, it is

$$\lambda \in \mathrm{Hom}_{\mathcal{O}}(V, k_x) \approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

so that for all indices i

$$\lambda(\mu_i) = f_i \mod x\mathcal{O}_x$$

A solution λ to the pointwise system Ξ_y at y is **holomorphic** at x (where $y \subset x$) if

$$\lambda(V \otimes_{\mathcal{O}} k_y) \subset \mathcal{O}_x / y\mathcal{O}_x$$

In that case, the solution λ to Ξ_y gives a solution to the pointwise system Ξ_x by taking the image of λ under the natural map

$$\mathrm{Hom}_{k_y}(V \otimes_{\mathcal{O}} k_y, \mathcal{O}_x/y\mathcal{O}_x) \approx \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}_x/y\mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{O}}(V, k_x) \approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

where the last map arises from the quotient map

$$\mathcal{O}_x/y\mathcal{O}_x \to \mathcal{O}_x/x\mathcal{O}_x = k_x$$

The following lemma is usually assimilated into the physical intuition of algebraic geometry, but we should pay attention to what goes into its proof, since the sense of our notion of *meager set* depends upon the truth of this lemma.

Lemma: Suppose that $\mathcal{O}$ is Noetherian. Given $f \neq 0$ in $\mathcal{M}$, there is a finite collection $\eta_1, \ldots, \eta_n$ of irreducible hyperplanes so that $1/f \in \mathcal{O}_x$ for x not in the union $\eta_1 \cup \ldots \cup \eta_n$. In particular, the hyperplanes are those attached to the isolated primes in a primary decomposition of the ideal $f\mathcal{O}$.

Proof: For a prime ideal $x \in X$, if $f \notin x$ then $1/f \in \mathcal{O}_x$. Thus, we must show that if $f \in x$ then there is a height-one prime y so that $f \in y \subset x$.

Let $J = \bigcap_i Q_i$ be a primary decomposition of an ideal $J \subset x$, where Q_i is primary with associated prime x_i. Taking radicals,

$$x = \mathrm{rad}\, x \supset \bigcap_i \mathrm{rad}\, Q_i = \bigcap_i x_i$$

so, for some index i, $x \supset x_i$. Thus, x must contain some one of the minimal (i.e., isolated) primes among the primes associated to J. Krull's Principal Ideal Theorem asserts that every prime *minimal* among those occurring in a primary decomposition of $f\mathcal{O}_x$ is height-one. This gives the result. ///

Lemma: Let Y be a non-meager subset of X, the prime spectrum of $\mathcal{O}$, where $\mathcal{O}$ is a Noetherian integral domain. Then

$$\bigcap_{x \in Y} x\mathcal{O}_x = \{0\}$$

Proof: Let $r \neq 0$ be in the indicated intersection. By the previous lemma, there would be finitely-many hypersurfaces $\eta_1, \ldots, \eta_n$ so that for $x \in X$ *off* the union of these hypersurfaces we would have both $r \in \mathcal{O}_x$ and $1/r \in \mathcal{O}_x$. Removing from Y the intersections of Y with these hypersurfaces would still leave a non-meager set Y_r. In particular, Y_r would be non-empty, and, by construction, for all $x \in Y_r$ both r and $1/r$ would lie in $\mathcal{O}_x$. But this would contradict the hypothesis $r \in x\mathcal{O}_x$. Thus, it must be that the indicated intersection is $\{0\}$, as claimed. ///

Lemma: Let $\Xi = \{(\mu_i, f_i) : i \in I\}$ be a $\mathcal{O}$-parametrized system with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$. Then
- There is a union $\bigcup_{i \in I} \eta_i$ of irreducible hyperplanes η_i off which the system Ξ is holomorphic.
- Let $\{m_j : j \in J\}$ be a generating set for V over $\mathcal{O}$. For a generic solution λ of Ξ, there is a union $\bigcup_{j \in J} \eta_i$ of irreducible hyperplanes η_i off which λ is holomorphic.

Proof: Let $\{g_i : i \in I\}$ be a collection of non-zero elements of $\mathcal{O}$ so that, for for all i, $g_i\mu_i \in M$ and $g_i f_i \in \mathcal{O}$. Then Ξ is holomorphic off the union of the hyperplanes attached to the isolated primes occurring in the primary decompositions of the ideals $g_i\mathcal{O}$.

Similarly, let $\{g_j : i \in J\}$ be a collection of non-zero elements of $\mathcal{O}$ so that, for for all $j \in J$, $g_i\lambda(m_j) \in \mathcal{O}$. Then λ is holomorphic off the union of the hyperplanes attached to the isolated primes occurring in the primary decompositions of the ideals $g_j\mathcal{O}$. ///

Theorem: Let k be a field, and $\mathcal{O}$ a commutative Noetheria k-algebra. Let $\Xi = \{(\mu_i, f_i) : i \in I\}$ be an $\mathcal{O}$-parametrized linear system with coefficients in $V \otimes_{\mathcal{O}} \mathcal{M}$ where V is a countably-generated $\mathcal{O}$-module, where also the index set I is countable.

- If there is a unique generic solution λ to Ξ, then for $x \in X$ off a meager set there is a unique solution λ_x to the pointwise linear system Ξ_x at x, and this pointwise solution is obtained from λ via the natural map

$$\mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{O}}(V, \mathbf{K}_x)$$

$$\approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

- Suppose that Y is a *non-meager* subset of X so that for $x \in Y$ the system Ξ is holomorphic at x, and so that for $x \in Y$ the pointwise system Ξ_x at x has a unique solution λ_x. Then there is a unique *generic solution* to Ξ.

Proof: First, we reduce to the case that at most one of the f_i is not $0 \in \mathcal{M}$. If all f_i are already 0, we are done. So suppose that some f_{i_o} is non-zero. For another index i, replace the condition

$$\lambda(v_i) = f_i$$

by the condition

$$\lambda(v_i - \frac{f_i}{f_{i_o}} v_{i_o}) = 0$$

The collection of generic solutions is unchanged by such an adjustment, and off a meager subset of X this change gives pointwise systems equivalent to the original pointwise system Ξ_x. Thus, overlooking a meager subset of X, we can assume without loss of generality that for at most one index i_o is f_{i_o} not 0.

Also, we ignore the countably-many hypersurfaces on which the system Ξ fails to be holomorphic.

If there is a unique generic solution λ, then by the proposition the vectors μ_i span $V \otimes_{\mathcal{O}} \mathcal{M}$, and if f_{i_o} is not 0 then μ_{i_o} does not lie in the $\mathcal{M}$-span of the other vectors.

Let $v_1, v_2, \ldots$ be a countable collection of generators for the $\mathcal{O}$-module V. The images of these v_j span all the vectorspaces $V \otimes_{\mathcal{O}} k_x$ for $x \in X$. Then $v_1 \otimes 1, v_2 \otimes 1, \ldots$ is an $\mathcal{M}$-basis for $V \otimes_{\mathcal{O}} \mathcal{M}$. Express each $m_j \otimes 1$ as an $\mathcal{M}$-linear combination of the μ_i as

$$v_j \otimes 1 = \sum_i g_{ji}\, \mu_i$$

with $g_{ji} \in \mathcal{M}$. Off the meager set of points $x \in X$ where some one of the g_{ji} fails to be holomorphic, tensoring with $k_x = \mathcal{O}_x / x\mathcal{O}_x$ expresses the image of v_j as a k_x-linear combination of the images of μ_i in $V \otimes_{\mathcal{O}} k_x$. (We have already excluded the meager set on which any one of the μ_i fails to be holomorphic). That is, we conclude that off a meager set the pointwise system Ξ_x has at most one solution.

Then, given a generic solution λ, off the meager set where λ fails to be holomorphic, the image of λ under the natural map

$$\mathrm{Hom}_{\mathcal{M}}(V \otimes_{\mathcal{O}} \mathcal{M}, \mathcal{O}_x) \approx \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}_x) \to \mathrm{Hom}_{\mathcal{O}}(V, k_x) \approx \mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$$

is a solution of the pointwise system Ξ_x. Thus, *uniqueness of the generic solution implies uniqueness of solution to pointwise systems Ξ_x for x off a meager set.*

On the other hand, suppose that for x in a non-meager subset Y (on which Ξ is holomorphic) the pointwise system Ξ_x has a unique solution. To prove that a generic solution exists, we must show that μ_{i_o} does not lie in the $\mathcal{M}$-span of the other μ_i. Indeed, if

$$\mu_{i_o} = \sum_{i \neq i_o} g_i \mu_i$$

with $g_i \in \mathcal{M}$, then, off the meager set where some one of the g_i or μ_i fails to be holomorphic, the image of μ_{i_o} in $V \otimes \mathcal{O} k_x$ would be a k_x-linear combination of the images of the other elements μ_i with $i \neq i_o$. The proposition above assures that this cannot happen. Thus, there is at least one generic solution.

Suppose that there were two distinct generic solutions, and call their difference δ. The pointwise uniqueness hypothesis assures that, for x in a non-meager set Y (on which we may assume δ holomorphic), the image of δ in $\mathrm{Hom}_{k_x}(V \otimes_{\mathcal{O}} k_x, k_x)$ is 0. Thus, for all $x \in Y$, for all of the countably-many $\mathcal{O}$-generators m_j for V,

$$\delta(m_j \otimes 1) \subset x\mathcal{O}_x$$

But we saw above that the intersection of sets $x\mathcal{O}_x$ for x ranging over any non-meager set is $\{0\}$. Thus, $\delta(m_j \otimes 1) = 0$, proving uniqueness. ///