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Discrete spectrum of Laplacians on compact manifolds

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Discrete spectrum of Laplacians on compact manifolds

Let M be a compact Riemannian manifold, thus equipped with Laplacian $\Delta = \Delta^M$ and a measure so that Δ is *symmetric* on $L^2(M) \cap C_c^\infty(M)$. For simplicity of notation, consider two-dimensional M.

Cover M with coordinate neighborhoods $\{V_m : m \in M\}$. At each $m \in M$ choose a smaller coordinate neighborhood U_m such that $\overline{U}_m \subset V_m$. Invoke compactness to produce a finite subcover $\{U_i = U_{m_i}\}$. Fix a smooth partition of unity $\{\varphi_i\}$ subordinate to that finite cover. Let $\psi_i : U_i \to \mathbb{R}^2$ be the coordinate map.

On $\psi_i(U_i)$, the image of the measure from M can be described by a two-form $\mu_i(x,y)\,dx\wedge dy$ with continuous $\mu_i > 0$. The shrinking of the coordinate patches (above) ensures that μ_i extends continuously to the (compact) closure of $\psi_i(U_i)$, so is *bounded* above and away from 0.

That is, the image on $\psi_i(U_i)$ of the measure from M is bounded above and below by non-zero constant multiples of $dx \wedge dy$.

There is large-enough $r > 0$ such that each $\psi_i(U_i)$ sits inside the rectangle

$$R \;=\; \{(x,y) \in \mathbb{R}^2 : |x| \le r,\; |y| \le r\}$$

Thus, given $f \in C^\infty(M)$, each $\varphi_i f$ on U_i descends to a smooth function $(\varphi_i \cdot f) \circ \psi_i^{-1}$ on $\psi_i(U_i) \subset R$, which is identified with a smooth function on the two-torus $\mathbb{R}^2/r\mathbb{Z}^2$.

Without loss of generality, all functions here are $\mathbb{R}$-valued.

[0.1] Comparison of L^2-norms of functions and smooth truncations

On one hand, because $0 \le \varphi_i \le 1$, certainly

$$|\varphi_i f|_{L^2(M)} \;\le\; |f|_{L^2(M)}$$

On the other hand,

$$|f|^2_{L^2(M)} \;=\; \Big\langle \sum_i \varphi_i \cdot f,\, f\Big\rangle_{L^2(M)} \;=\; \sum_i \Big|\langle \varphi_i f,\, f\rangle_{L^2(M)}\Big| \;\le\; \sum_i |\varphi_i f|_{L^2(M)} \cdot |f|_{L^2(M)}$$

Cancelling the factor of $|f|$ from both sides,

$$|f|_{L^2(M)} \;\le\; \sum_i \big|\varphi_i f\big|_{L^2(M)}$$

[0.2] Comparison of H^1-norms of functions and smooth truncations

A similar argument gives one direction of bound:

$$|f|^2_{H^1(M)} \;=\; \Big\langle \sum_i \varphi_i f, f\Big\rangle_{H^1(M)} \;=\; \sum_i \langle \varphi_i f, f\rangle_{H^1M} \;\le\; \sum_i |\varphi_i f|_{H^1(M)} \cdot |f|_{H^1(M)}$$

so

$$|f|_{H^1M} \;\le\; \sum_i |\varphi_i f|_{H^1(M)}$$

From the other side, integrating by parts, now denoting the pairing $\langle,\rangle_m$ in the tangent space to M at m by $v \cdot w$ and writing $\|v\| = (v\cdot v)^{\frac{1}{2}}$,

$$|\varphi_i f|^2_{H^1(M)} \;\le\; \int_M (-\Delta+1)(\varphi_i f)\,\varphi_i f \;=\; \int_M \nabla(\varphi_i f)\cdot\nabla(\varphi_i f) + \int_M \varphi_i f\,\varphi_i f$$

$$= \int_M \big(f\nabla\varphi_i + \varphi_i \nabla f\big)\cdot\big(f\nabla\varphi_i + \varphi_i\nabla f\big) + |\varphi_i f|^2_{L^2} \;=\; \int_M f^2 \|\nabla\varphi_i\|^2 + \int_M 2f\varphi_i \nabla f\cdot\nabla\varphi_i + |\varphi_i f|^2_{L^2}$$

The first and last summands are dominated by $|f|^2_{L^2}$ with an implied constant independent of f. For the middle term, by Cauchy-Schwarz-Bunyakowsky,

$$\Big|\int_M 2f\varphi_i\nabla f\cdot\nabla\varphi_i\Big| \;\le\; \int_M 2\varphi_i\;|f|\;\|\nabla f\|\;\|\nabla\varphi_i\| \;\ll\; \int_M |f|\;\|\nabla f\| \;\le\; \Big(\int_M |f|^2\Big)^{\frac12}\Big(\int_M \|\nabla f\|^2\Big)^{\frac12}$$

$$= |f|_{L^2}\cdot\Big(\int_M -\Delta f\; f\Big)^{\frac12} \;\le\; |f|_{L^2}\cdot\Big(\int_M (1-\Delta)f\; f\Big)^{\frac12} \;=\; |f|_{L^2}\cdot|f|_{H^1} \;\le\; |f|^2_{H^1(M)}$$

That is, with an implied constant independent of f,

$$|\varphi_i f|_{H^1(M)} \;\ll\; |f|_{H^1(M)}$$

[0.3] Comparison to flat-tori norms Thus, we consider f supported in a single coordinate patch U, viewed as sitting inside the rectangle R, which we map to a two-torus $\mathbb{T}^2$ by identifying opposite sides. Smooth functions supported on U descend to smooth functions on $\mathbb{T}^2$. Suppress the index i, view the coordinate map $\psi = \psi_i$ be an inclusion, and suppress ψ from the notation. It is easy to compare the $L^2(M)$-norm of such f to the flat-torus L^2-norm:

$$|f|^2_{L^2(M)} \;=\; \int_R |f|^2\;\mu(x,y)\,dx\,dy \;\ll\; \int_R |f|^2\;dx\,dy \;=\; |f|^2_{L^2(\mathbb{T}^2)} \qquad (\mu\text{ bounded above})$$

Conversely,

$$|f|^2_{L^2(\mathbb{T}^2)} \;=\; \int_R |f|^2\;dx\,dy \;\ll\; \int_R |f|^2\;\mu(x,y)\;dx\,dy \;=\; |f|^2_{L^2(M)} \qquad (\mu\text{ bounded below})$$

For the H^1-norm, integrating by parts on M,

$$\int_M -\Delta^M f\cdot f \;=\; \int_M \nabla^M f\cdot\nabla^M f \;=\; \int_R (af_x + bf_y)^2 + (cf_x + df_y)^2\;\mu(x,y)\;dx\,dy$$

for some smooth coefficient functions a,b,c,d. On one hand, this is clearly dominated by $\int_R (f_x)^2 + (f_y)^2\;dx\,dy$. On the other hand, the ellipticity of Δ^M promises that the quadratic forms

$$Q(u,v) \;=\; (au+bv)^2 + (cu+du)^2 \;=\; (a^2+c^2)u^2 + 2(ab+cd)uv + (b^2+d^2)v^2$$

have $a^2+c^2>0$ and $b^2+d^2>0$ uniformly on U, and the discriminant is uniformly negative on U. That is,

$$u^2+v^2 \;\ll\; Q(u,v) \;\ll\; u^2+v^2 \qquad (\text{uniformly on } U)$$

Thus,

$$\int_R (f_x)^2 + (f_y)^2 \;\ll\; \int_M -\Delta f\cdot f \;\ll\; \int_R (f_x)^2 + (f_y)^2$$

and

$$|f|_{H^1(R)} \;\ll\; |f|_{H^1(M)} \;\ll\; |f|_{H^1(R)}$$

With f descended to a smooth function on $\mathbb{T}^2$, this is

$$|f|_{H^1(\mathbb{T}^2)} \;\ll\; |f|_{H^1(M)} \;\ll\; |f|_{H^1(\mathbb{T}^2)}$$

[0.4] Compactness of $H^1(M) \to L^2(M)$ Let V_i^o be the closure in $L^2(\mathbb{T}^2)$ of $\{\varphi_i f : f \in L^2(M)\}$, and let V_i^1 be the closure in $H^1(\mathbb{T}^2)$ of $\{\varphi_i f \circ \psi_i^{-1} : f \in H^1(M)\}$.

The exponentials $\psi_\xi(x) = e^{\pi i \langle x, \xi > / r}$ form an orthogonal basis in both Hilbert spaces, but $|\psi_\xi|_{H^1} = \|\xi\| \cdot |\psi_\xi|_{L^2}$ where $\|\xi\|$ is the Euclidean norm of $\xi \in \mathbb{Z}^2$. Since $\|\xi\| \to +\infty$, this proves a simple Rellich lemma: $H^1(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is *compact*.

As a corollary, the restriction to $V_i^1 \to V_i^o$ is compact.

The estimates above demonstrate continuity of $H^1(M) \to \bigoplus_i V_i^1$ and $\bigoplus_i V_i^o \to L^2(M)$ given by

$$f \to \{\varphi_i f \circ \psi_i^{-1}\} \qquad \text{and} \qquad \{g_i \in L^2(\psi_i(U_i))\} \longrightarrow \sum_i g_i \circ \psi_i$$

Thus, the composite

$$H^1(M) \longrightarrow \bigoplus_i V_i^1 \longrightarrow \bigoplus_i V_i^o \longrightarrow L^2(M)$$

is compact.

[0.5] Discreteness of spectrum of Δ^M Since the resolvent of the Friedrichs extension $\tilde{\Delta}$ of $\Delta = \Delta^M$ maps continuously $L^2(M) \to H^1(M)$ and $H^1(M) \to L^2(M)$ is compact, the resolvent is compact, so has purely discrete spectrum. The eigenfunctions for the resolvent are those of $\tilde{\Delta}$, so $L^2(M)$ has a basis of $\tilde{\Delta}$-eigenfunctions.

In fact, $\tilde{\Delta}$-eigenfunctions are in $H^\infty(M)$. Local versions of the standard Sobolev inequalities/imbeddings, in effect on $\mathbb{T}^2$, show that $H^\infty(M) = C^\infty(M)$, so $\tilde{\Delta}$-eigenfunctions are C^∞, and evaluation of $\tilde{\Delta}$ on them is simply evaluation of Δ. Thus, the spectrum of Δ is purely discrete.
