(*January 15, 2015*)

Satake parameters versus unramified principal series

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[From 1999. Edited 2015.]

We show that the *Satake parameters* attached to a spherical representation π via Satake's isomorphism (1963, IHES) can also be evaluated via the character χ into whose associated unramified principal series the spherical representation π imbeds. This seems to be "well-known", but also apocryphal.

Let G be a reductive p-adic group defined over an ultrametric local field k. Let P be a minimal parabolic (defined over k), with unipotent radical N and choice of Levi component M. For all of these groups G, P, N, M, use the symbols G, P, N, M to refer to their k-valued points.

With $\mathscr{N}$ the normalizer of M in G, the **(spherical) Weyl group** W is

$$W = \mathscr{N}/M$$

Let K be a *special* maximal compact subgroup of G. The **spherical Hecke algebra** $H_{G,K}$ of G (with respect to K) is

$$H_{G,K} = \{\text{left and right } K\text{-invariant } \mathbb{C}\text{-valued compactly-supported functions } f \text{ on } G\}$$

The subgroup

$$M_o = M \cap K$$

is the *unique* maximal compact subgroup of M, is *normal* in M, and

$$M/M_o \approx \mathbb{Z}^r$$

where r is the k-rank of G (and of M). The spherical Hecke algebra H_{M,M_o} of M with respect to M_o is acted upon by the Weyl group W in the obvious manner: for $f \in H_{M,M_o}$, $w \in W$, $m \in M$,

$$f^w(m) = f(w^{-1}\, m\, w)$$

where $w^{-1}\, m\, w$ is computed in M/M_o by replacing w by a pre-image of it in $\mathscr{N}$. Let

$$H^W_{M,M_o} = \{\ W\text{-invariant elements of } H_{M,M_o}\ \}$$

The **Satake transform**

$$S : H_{G,K} \longrightarrow H_{M,M_o}$$

is

$$(Sf)(m) = \delta(m)^{-1/2} \int_N f(nm)\, dn = \delta(m)^{1/2} \int_N f(mn)\, dn$$

where $\delta(m)$ is the modular function

$$\delta(m) \cdot dn = d(mnm^{-1})$$

with Haar measure dn on (unimodular) N. **Satake's theorem** is that S maps to the W-invariant subalgebra H^W_{M,M_o}, and is an *isomorphism*

$$S : H_{G,K} \approx H^W_{M,M_o}$$

The quotient M/M_o is isomorphic to $\mathbb{Z}^r$, and the full spherical Hecke algebra H_{M,M_o} is a finitely-generated commutative $\mathbb{C}$-algebra. For example, for $G = GL_n$ it is $\mathbb{C}[x_1, \ldots, x_n]$ and for $G = Sp_n$ is is

$\mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$. For $G = GL(n)$, the Weyl group W is isomorphic to the symmetric group on n letters, permuting generators mapping to x_i. For Sp_n, W is isomorphic to *signed* permutations on n letters.

Satake further observes that H_{M,M_o} is *integral* over the Weyl-group-invariant subring H^W_{M,M_o}, so H^W_{M,M_o} is commutative and Noetherian itself, and every algebra homomorphism

$$\lambda' : H^W_{M,M_o} \longrightarrow \mathbb{C}$$

extends (not uniquely) to an algebra homomorphism of the *full* spherical Hecke algebra

$$\tilde{\lambda}' : H_{M,M_o} \longrightarrow \mathbb{C}$$

Thus, by Satake's isomorphism, the spherical Hecke algebra $H_{G,K}$ of G is *Noetherian* and *commutative*. Commutativity is more elementary for classical groups, but Noetherian-ness is more substantial.

A **(K-) spherical representation** of G is an irreducible smooth representation π of G with a non-zero K-fixed vector v_o. A K-fixed vector in *any* representation of G is called a **spherical vector**. Using the commutativity of $H_{G,K}$, the subspace of all K-fixed vectors in an irreducible representation is at most one-dimensional. Thus, in a spherical representation the subspace of spherical vectors is exactly one-dimensional.

A spherical representation π of G gives an algebra homomorphism

$$\lambda_\pi : H_{G,K} \longrightarrow \mathbb{C}$$

by its action on a non-zero spherical vector:

$$\pi(\eta)v_o = \lambda_\pi(\eta) \cdot v_o$$

where the action of $H_{G,K}$ on the representation space of π is the usual

$$\pi(\eta)v = \int_G \eta(g)\, \pi(g)v\, dg$$

Satake's isomorphism S has an inverse S^{-1}, so for any such λ_π, there is the composition

$$\lambda'_\pi = \lambda_\pi \circ S^{-1} : H^W_{M,M_o} \longrightarrow \mathbb{C}$$

which extends (not uniquely) to an algebra homomorphism

$$\tilde{\lambda}'_\pi : H_{M,M_o} \longrightarrow \mathbb{C}$$

Depending upon the choice of generators m_i for the quotient M/M_o, the **Satake parameters** attached to π are the images $\tilde{\lambda}'_\pi(\mathrm{ch}_{m_1 M_o}), \ldots, \tilde{\lambda}'_\pi(\mathrm{ch}_{m_r M_o})$ of characteristic functions $\mathrm{ch}_{m_i M_o}$ of the sets $m_i M_o$.

An algebra homomorphism

$$\mu : H_{M,M_o} \longrightarrow \mathbb{C}$$

gives rise to an M_o-spherical representation $\sigma = \sigma_\mu$ of M, which by Schur's Lemma and the abelian-ness of M/M_o is necessarily one-dimensional, given by

$$\sigma_\mu(m) = \mu(mM_o)$$

This σ is *unramified*, meaning that it is trivial on M_o. Further, since σ is one-dimensional, it would be referred to simply as an *unramified character*.

Summing up, a spherical representation π of G gives rise to an algebra homomorphism

$$\lambda_\pi : H_{G,K} \longrightarrow \mathbb{C}$$

which by Satake's isomorphism gives an algebra homomorphism

$$\lambda'_\pi : H^W_{M,M_o} \longrightarrow \mathbb{C}$$

which extends to an algebra homomorphism

$$\tilde{\lambda'}_\pi : H_{M,M_o} \longrightarrow \mathbb{C}$$

which gives rise to an unramified character

$$\sigma_\pi = \sigma_{\tilde{\lambda}'_\pi} : M \longrightarrow M/M_o \longrightarrow \mathbb{C}^\times$$

Since the extension $\tilde{\lambda'_\pi}$ is ambiguous by the action of W, σ_π is likewise ambiguous.

On the other hand, from the theorem of Borel-Casselman-Matsumoto, a spherical representation π has an injection

$$\pi\, To\, \mathrm{Ind}_P^G\, \chi\delta^{1/2}$$

to an unramified principal series $\mathrm{Ind}_P^G\, \chi\delta^{1/2}$, meaning that the character $\chi = \chi_\pi$ on M is trivial on M_o (and is extended to $P = MN$ by being required to be trivial on N).

The Weyl group acts upon unramified characters of M by

$$\chi^w(m) = \chi(wmw^{-1})$$

For χ *generic* (in a sense which does not concern us too much here), the corresponding unramified principal series is *irreducible*, and

$$\mathrm{Ind}_P^G\, \chi^w\delta^{1/2} \approx \mathrm{Ind}_P^G\, \chi\delta^{1/2}$$

Thus, generically, the choice of unramified principal series into which a spherical representation imbeds is ambiguous by elements of W, and the unramified character χ_π is likewise ambiguous.

Thus, to a spherical representation π of G we have attached two unramified characters, σ_π and χ_π, both of which are usually ambiguous by action of W.

Small Apocryphal Theorem: The two characters associated above to a spherical representation π are the same (modulo the action of the spherical Weyl group W). That is, in the notation above, and modulo the action of W,

$$\chi_\pi = \sigma_\pi$$

Proof: Imbed the spherical representation π in an unramified principal series $i_\chi = \mathrm{Ind}_P^G\, \chi\delta^{1/2}$. Let φ be the canonical spherical vector in this unramified principal series, namely

$$\varphi(pk) = \varphi(p) = (\chi\delta^{1/2})(p)$$

for $p \in P$ and $k \in K$, using a p-adic Iwasawa decomposition $G = PK$. Also by an Iwasawa decomposition, the vectorspace of K-spherical vectors in this unramified principal series is one-dimensional. Thus, φ spans the subspace of spherical vectors. Thus, for $\eta \in H_{G,K}$,

$$i_\chi(\eta)\,\varphi = \lambda_\pi(\eta)\cdot\varphi$$

for the algebra homomorphism λ_π attached to π. Since the action here is explicit, by the right regular representation, we can express this as an integral:

$$i_\chi(\eta)\,\varphi(g) \;=\; \lambda_\pi(\eta)\cdot\varphi(g) \;=\; \int_G \eta(h)\cdot\varphi(gh)\,dh$$

To determine $\lambda_\pi(\eta)$, since $\varphi(1)=1$, it suffices to compute the integral when $g=1$. Thus,

$$\lambda_\pi(\eta) \;=\; i_\chi(\eta)\,\varphi(1) \;=\; \int_G \eta(h)\cdot\varphi(h)\,dh$$

It is an exercise to show that (up to normalizing constant), for any compactly-supported complex-valued measuable function f on G

$$\int_G f(g)\,dg \;=\; \int_P\int_K f(p^{-1}k)\,dp\,dk$$

where both measures are *right* Haar measures. Replacing p by p^{-1} transforms this to

$$\int_G f(g)\,dg \;=\; \int_P\int_K f(pk)\,\delta(p)^{-1}\,dp\,dk$$

where again (as above) δ is the modular function on P. Thus,

$$\lambda_\pi(\eta) \;=\; \int_G \eta(h)\cdot\varphi(h)\,dh \;=\; \int_P\int_K \eta(pk)\,\varphi(pk)\,\delta(p)^{-1}\,dp\,dk$$

Normalizing the measure on K to be 1, using the right K-invariance of the integrand,

$$\lambda_\pi(\eta) \;=\; \int_G \eta(h)\cdot\varphi(h)\,dh \;=\; \int_P \eta(p)\,\varphi(p)\,\delta(p)^{-1}\,dp$$

Restricted to P, φ is just $\chi\delta^{1/2}$, so this is

$$\lambda_\pi(\eta) \;=\; \int_P \eta(p)\,\chi\delta^{1/2}(p)\,\delta(p)^{-1}\,dp \;=\; \int_P \eta(p)\,\chi\delta^{-1/2}(p)\,dp$$

Break up the Haar measure on P in terms of the Haar measures on M and N, with $P=NM$: for suitable function f on P,

$$\int_P f(p)\,dp \;=\; \int_M\int_N f(nm)\,dn\,dm$$

The order in $f(nm)$ *does* matter. Then

$$\lambda_\pi(\eta) \;=\; \int_M\int_N \eta(nm)\,\chi\delta^{-1/2}(nm)\,dn\,dm \;=\; \int_M\int_N \eta(nm)\,\chi\delta^{-1/2}(m)\,dn\,dm$$

$$=\; \int_M \chi(m)\cdot\Big(\delta^{-1/2}(m)\int_N \eta(nm)\;dn\Big)\,dm \;=\; \int_M \chi(m)\,(S\eta)(m)\,dm$$

That is, σ_π can be evaluated on images $S\eta$ of elements η of $H_{G,K}$ under the Satake map by using the same character χ_π that occurs in an unramified principal series i_χ into which π imbeds. This is what was to be proven. ///