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Self-dualities

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For an abelian (locally compact, Hausdorff) topological group G, let $G^\vee$ be the *unitary dual*, that is, the collection of continuous group homomorphisms of G to the unit circle in $\mathbb{C}^\times$. For *compact totally disconnected* G, since $\mathbb{C}^\times$ contains no *small subgroups*, every element of $G^\vee$ has image in roots of unity in $\mathbb{C}^\times$, which can be identified with $\mathbb{Q}/\mathbb{Z}$. Thus, for compact totally disconnected G,

$$G^\vee \;\approx\; \mathrm{Hom}^o(G,\,\mathbb{Q}/\mathbb{Z}) \qquad\qquad \text{(continuous homomorphisms)}$$

where $\mathbb{Q}/\mathbb{Z} = \mathrm{colim}\,\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ is *discrete*. As a topological group, $\mathbb{Z}_p = \lim \mathbb{Z}/p^\ell\mathbb{Z}$. It is also useful to observe that $\mathbb{Z}_p$ is a limit of the corresponding quotients of itself, namely,

$$\mathbb{Z}_p \;\approx\; \lim \mathbb{Z}_p/p^\ell\mathbb{Z}_p$$

Indeed, more generally, every abelian *totally disconnected* topological group G has the property that

$$G \;\approx\; \lim_K G/K$$

where K ranges over compact open subgroups of G. Also, as a topological group,

$$\mathbb{Q}_p \;=\; \bigcup \frac{1}{p^\ell}\mathbb{Z}_p \;=\; \mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p$$

Because of the *no small subgroups* property of the unit circle in $\mathbb{C}^\times$, every continuous element of $\mathbb{Z}_p^\vee$ factors through *some* limitand

$$\mathbb{Z}_p/p^\ell\mathbb{Z}_p \;\approx\; \mathbb{Z}/p^\ell\mathbb{Z}$$

Thus,

$$\mathbb{Z}_p^\vee \;=\; \mathrm{colim}\,\Big(\mathbb{Z}_p/p^\ell\mathbb{Z}_p\Big)^\vee \;=\; \mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$$

since $\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p$ is the dual to $\mathbb{Z}_p/p^\ell\mathbb{Z}_p$ under the pairing

$$\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\times\; \mathbb{Z}_p/p^\ell\mathbb{Z}_p \;\approx\; \frac{1}{p^\ell}\mathbb{Z}/\mathbb{Z} \;\times\; \mathbb{Z}/p^\ell\mathbb{Z} \;\ni\; \big(\frac{x}{p^\ell}+\mathbb{Z}\big) \;\times\; \big(y+p^\ell\mathbb{Z}\big) \;\longrightarrow\; xy+\mathbb{Z} \;\in\; \mathbb{Q}/\mathbb{Z}$$

The transition maps in the colimit expression for $\mathbb{Z}_p^\vee$ are inclusions, so

$$\mathbb{Z}_p^\vee \;=\; \mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p/\mathbb{Z}_p \;\approx\; \Big(\mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)/\mathbb{Z}_p \;\approx\; \mathbb{Q}_p/\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \Big(\mathrm{colim}\,\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \lim(\frac{1}{p^\ell}\mathbb{Z}_p)^\vee$$

As a topological group, $\frac{1}{p^\ell}\mathbb{Z}_p \approx \mathbb{Z}_p$ by multiplying by p^ℓ, so the dual of $\frac{1}{p^\ell}\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p^\vee \approx \mathbb{Q}_p/\mathbb{Z}_p$. However, the inclusions for varying ℓ are not the identity map, so for compatibility take

$$\Big(\frac{1}{p^\ell}\mathbb{Z}_p\Big)^\vee \;=\; \mathbb{Q}_p/p^\ell\mathbb{Z}_p$$

Thus,

$$\mathbb{Q}_p^\vee \;=\; \lim \mathbb{Q}_p/p^\ell\mathbb{Z}_p \;\approx\; \mathbb{Q}_p$$

because, again, any abelian totally disconnected group is the projective limit of its quotients by compact open subgroups.

The same argument applies to $\widehat{\mathbb{Z}} = \lim \mathbb{Z}/N\mathbb{Z}$ and finite adeles $\mathbb{A}_{\rm fin} = \mathrm{colim}\,\frac{1}{N}\widehat{\mathbb{Z}}$, proving the self-duality of $\mathbb{A}_{\rm fin}$. [1] That is, $\widehat{\mathbb{Z}}^\vee \approx \mathbb{A}_{\rm fin}/\widehat{\mathbb{Z}}$, and so on.

Similarly, the same argument applies over an arbitrary finite extension k_v of $\mathbb{Q}_p$, but now the pairing is composed with the local *trace* from k_v to $\mathbb{Q}_p$ and the dual lattice to the local integers $\mathfrak{o}_v$ is (by definition) the *inverse different* $\mathfrak{d}_v^{-1}$, in general strictly larger than the local integers. Let's execute the argument:

Let $\mathfrak{m}_v$ be the maximal ideal in $\mathfrak{o}_v$. As a topological group, $\mathfrak{o}_v = \lim \mathfrak{o}/\mathfrak{p}^\ell$, for any number field k giving rise to the local field extension $k_v/\mathbb{Q}_p$, and k having integers $\mathfrak{o}$. However, we do not need to refer to any global object, as the question is local. That is, more to the point, $\mathfrak{o}_v$ is a limit of the corresponding quotients of itself,

$$\mathfrak{o}_v \;\approx\; \lim \mathfrak{o}_v/\mathfrak{m}_v^\ell$$

Also, as a topological group,

$$k_v \;=\; \bigcup \mathfrak{m}_v^{-\ell} \;=\; \mathrm{colim}\,\mathfrak{m}_v^{-\ell}$$

Every continuous element of $\mathfrak{o}_v^\vee$ factors through *some* limitand, so

$$\mathfrak{o}_v^\vee \;=\; \mathrm{colim}\,\Big(\mathfrak{o}_v/\mathfrak{m}_v^\ell\Big)^\vee \;=\; \mathrm{colim}\,\mathfrak{m}_v^{-\ell}/\mathfrak{d}_v^{-1}$$

since $\mathfrak{m}_v^{-\ell}/\mathfrak{d}_v^{-1}$ is the dual to $\mathfrak{o}_v/\mathfrak{m}_v^\ell$ under the pairing

$$\mathfrak{m}_v^{-\ell}/\mathfrak{d}_v^{-1} \;\times\; \mathfrak{o}_v/\mathfrak{m}_v^\ell \;\ni\; \big(x+\mathfrak{d}_v^{-1}\big) \;\times\; \big(y+\mathfrak{m}_v^\ell\big) \;\longrightarrow\; xy+\mathfrak{d}_v^{-1} \;\longrightarrow\; \mathrm{tr}_{k_v/\mathbb{Q}_p}(xy)+\mathbb{Z}_p \;\in\; \mathbb{Q}_p/\mathbb{Z}_p \;\subset\; \mathbb{Q}/\mathbb{Z}$$

by additive approximation.

The transition maps in the colimit expression for $\mathfrak{o}_v^\vee$ are inclusions, so

$$\mathfrak{o}_v^\vee \;=\; \mathrm{colim}\,\mathfrak{m}_v^{-\ell}/\mathfrak{d}_v^{-1} \;\approx\; \Big(\mathrm{colim}\,\mathfrak{m}_v^{-\ell}\Big)/\mathfrak{d}_v^{-1} \;\approx\; k_v/\mathfrak{d}_v^{-1}$$

Thus,

$$k_v^\vee \;=\; \Big(\mathrm{colim}\,\mathfrak{m}_v^{-\ell}\Big)^\vee \;=\; \lim(\mathfrak{m}_v^{-\ell})^\vee$$

As a topological group, $\mathfrak{m}_v^{-\ell}$ is non-canonically isomorphic to $\mathfrak{o}_v$ by multiplying by a power of a local parameter, so the dual of $\mathfrak{m}_v^\ell$ is isomorphic to $\mathfrak{o}_v^\vee \approx k_v/\mathfrak{d}_v^{-1}$. However, these isomorphisms are not *natural*, and, commensurately, the inclusions for varying ℓ are *not* identity maps, so for compatibility take

$$\Big(\mathfrak{m}_v^\ell\Big)^\vee \;=\; k_v/\mathfrak{m}_v^\ell\mathfrak{d}_v^{-1}$$

Thus,

$$k_v^\vee \;=\; \lim k_v/\mathfrak{m}_v^\ell\mathfrak{d}_v^{-1} \;\approx\; k_v$$

because an abelian totally disconnected group is the limit of its quotients by compact open subgroups.

[1] The traditional notation $\widehat{\mathbb{Z}}$ does also refer to $\mathrm{Hom}^o(\mathbb{Z},\mathbb{Q}/\mathbb{Z})$, but is often thought of differently, and needs to be topologized.