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Simplest automorphic Schrödinger operators

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1. Compact resolvent of $-\Delta + q$ for $q \geq y^\varepsilon$
2. Mellin transform functionals in $\mathfrak{B}^{-1}$
3. Hilbert-Schmidt resolvent of $-\Delta + q$ for $q \gg y^2$.

Evaluations of standard L-functions,

$$f \longrightarrow \Lambda(f,s) = \int_0^\infty y^{s-\frac{1}{2}} f(iy) \frac{dy}{y} \qquad \text{(for fixed } s \in \mathbb{C}, \text{ cuspform } f)$$

for cuspforms for $\Gamma = SL_2(\mathbb{Z})$, are *not* continuous functionals on $L^2(\Gamma\backslash\mathfrak{H})$. [1] We can try to remedy this by forming *global automorphic* Levi-Sobolev spaces

$$H^n(\Gamma\backslash\mathfrak{H}) = \text{Hilbert-space completion of } C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ under } |f|^2_{\mathfrak{B}^n(-\Delta+1)} = \langle(-\Delta+1)^n f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$$

with the usual

$$\Delta = y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

Compactly-supported automorphic distributions *do* lie in

$$H^{-\infty}(\Gamma\backslash\mathfrak{H}) = \text{colim}_n\, H^{-n}(\Gamma\backslash\mathfrak{H}) \qquad (H^{-n}(\Gamma\backslash\mathfrak{H}) \text{ the Hilbert-space dual of } H^n(\Gamma\backslash\mathfrak{H}))$$

However, $f \to \Lambda(f,s)$ does *not* lie in $H^{-\infty}(\Gamma\backslash\mathfrak{H})$.

A related problem is that the *continuous spectrum* of Δ entails that (the Friedrichs extension of) $-\Delta+1$ cannot have compact resolvent, so the injections $H^n(\Gamma\backslash\mathfrak{H}) \to H^{n-k}(\Gamma\backslash\mathfrak{H})$ are *not* compact, and $H^{+\infty}(\Gamma\backslash\mathfrak{H}) = \lim H^n(\Gamma\backslash\mathfrak{H})$ is *not* a nuclear Fréchet space. [2]

These and other issues are addressed by considering perturbations $-\Delta+q$ of $-\Delta$ by *potentials* $q \geq 1$ on $\Gamma\backslash\mathfrak{H}$ with *growth* at infinity, and corresponding generalized Levi-Sobolev spaces [3]

$$\mathfrak{B}^n = \text{Hilbert-space completion of } C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ under } |f|_{\mathfrak{B}^n(-\Delta+q)} = \langle(-\Delta+q)^n f, f\rangle_{L^2(\Gamma\backslash\mathfrak{H})}$$

The perturbation $-\Delta+q$ has *compact resolvent* for $q(x+iy) \gg y^\varepsilon$ for $\varepsilon > 0$. The L-function evaluation functionals $f \to \Lambda(f,s)$ *are* in the Hilbert-space dual $\mathfrak{B}^{-1}$ of $\mathfrak{B}^1$ on vertical strips $|\sigma - \frac{1}{2}| < \alpha$ for $q(x+iy) \gg y^{2\alpha}$. For $q(x+iy) \gg y^2$, the resolvent of $-\Delta+q$ is not merely compact, but is *Hilbert-Schmidt*, so $\mathfrak{B}^\infty = \lim_n \mathfrak{B}^n$ is *nuclear Fréchet*.

[1] In effect, [Good 1986] solves $(-\Delta+\lambda)u = \mu_s$ on $\mathfrak{H}$, with $\mu_s(f) = \int_0^\infty y^{s-\frac{1}{2}} f(iy)\, dy/y$ for $f \in C_c^\infty(\mathfrak{H})$, forms a Poincaré series $F_{s,w}$ from the free-space solution u, and meromorphically continues to obtain an automorphic form F_s such that $\int_{\Gamma\backslash\mathfrak{H}} f \cdot F_s = \Lambda(f,s)$. Analytic behavior of such Poincaré series is non-trivial.

[2] Projective limits of Hilbert spaces with Hilbert-Schmidt transition maps are the most important class of *nuclear Fréchet spaces*, by any definition, so we take this to be the definition of *nuclear Fréchet*. As usual, a chief application is existence of genuine *tensor products* of nuclear Fréchet spaces, from which a *Schwartz kernel theorem* follows almost immediately. E.g., see [Garrett 2012].

[3] This abstracted form of Levi-Sobolev spaces was considered at latest by the 1960s. For example, see [Pietsch 1966].

1. *Compact resolvent of* $-\Delta + q$ *for* $q \geq y^\varepsilon$

[1.0.1] **Theorem:** For a potential q with $q(x+iy) \gg y^\varepsilon$ in $y \geq 1$ for some $\varepsilon > 0$, the Friedrichs extension $\widetilde{S}$ of the Schrödinger operator

$$S \;=\; -\Delta + q$$

has *compact resolvent* $(\widetilde{S} - \lambda)^{-1}$, so has *discrete spectrum.*

Proof: The argument is an easier variant of the compactness argument in [Lax-Phillips 1976], p. 206. Let $\mathfrak{B}^1 = \mathfrak{B}^1(-\Delta + q)$. By construction, the inverse $\widetilde{S}^{-1}$ of the Friedrichs extension $\widetilde{S}$ of S maps continuously to $L^2(\Gamma\backslash\mathfrak{H}) \to \mathfrak{B}^1$, the latter topology finer than that of $L^2(\Gamma\backslash\mathfrak{H})$. Compactness of $\widetilde{S}^{-1} : L^2(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})$ would follow from compactness of the inclusion $\mathfrak{B}^1 \to L^2(\Gamma\backslash\mathfrak{H})$. Standard perturbation theory would prove that $(\widetilde{S} - \lambda)^{-1}$ exists (as bounded operator) for λ off a set with accumulation point at most 0, and is a compact operator there, and the spectrum of $\widetilde{S}$ is inverses of non-zero elements of the spectrum of $\widetilde{S}^{-1}$.

The *total boundedness* criterion for relative compactness requires that, given $\varepsilon > 0$, the image of the unit ball B in $\mathfrak{B}^1$ in $L^2(\Gamma\backslash\mathfrak{H})$ can be covered by finitely-many balls of radius ε.

The usual Rellich-Kondrachev compactness lemma, asserting compactness of injections $H^s(\mathbb{T}^n) \to H^t(\mathbb{T}^n)$ for $s > t$ of standard Levi-Sobolev spaces on products of circles, will reduce the issue to an estimate on the *tail* of $\Gamma\backslash\mathfrak{H}$, which will follow from the $\mathfrak{B}^1$ condition.

Given $c \geq 1$, cover the image Y_o of $\frac{\sqrt{3}}{2} \leq y \leq c+1$ in $\Gamma\backslash\mathfrak{H}$ by small coordinate patches U_i, and one large open U_∞ covering the image Y_∞ of $y \geq c$. Invoke compactness of Y_o to obtain a finite sub-cover of Y_o. Choose a smooth partition of unity $\{\varphi_i\}$ subordinate to the finite subcover along with U_∞, letting φ_∞ be a smooth function that is identically 1 for $y \geq c+1$. A function f in $\mathfrak{B}^1$ on Y_o is a finite sum of functions $\varphi_i \cdot f$. The latter can be viewed as having compact support on small opens in $\mathbb{R}^2$, thus identified with functions on products $\mathbb{T}^2$ of circles, and lying in $H^1(\mathbb{T}^2)$, since

$$\langle(-\Delta + q)\varphi_i f, \varphi_i f\rangle \;\ll_i\; \langle(-\Delta^E + 1)\varphi_i f, \varphi_i f\rangle \qquad \text{(with usual Euclidean Laplacian } \Delta^E)$$

The Rellich-Kondrachev lemma applies to each copy of the inclusion map $H^1(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$, so $\varphi_i \cdot B$ is totally bounded in $L^2(\Gamma\backslash\mathfrak{H})$.

Thus, to prove compactness of the global inclusion, it suffices to prove that, given $\delta > 0$, the cut-off c can be made sufficiently large so that $\varphi_\infty \cdot B$ lies in a single ball of radius δ inside $L^2(\Gamma\backslash\mathfrak{H})$. Since $0 \leq \varphi_\infty \leq 1$, it suffices to show

$$\lim_{c\to\infty} \int_{y>c} |f(z)|^2 \, \frac{dx\, dy}{y^2} \;\longrightarrow\; 0 \qquad \text{(uniformly for } |f|_{\mathfrak{B}^1} \leq 1)$$

We have

$$\int_{y>c} |f(z)|^2 \, \frac{dx\, dy}{y^2} \;\leq\; c^{-\varepsilon} \cdot \int_{y>c} |f(z)|^2 \, y^\varepsilon \, \frac{dx\, dy}{y^2}$$

$$\leq\; c^{-\varepsilon} \cdot \int_{y>c} |f(z)|^2 \, (-\Delta + y^\varepsilon) \, \frac{dx\, dy}{y^2} \;\leq\; c^{-\varepsilon} \;\longrightarrow\; 0 \qquad \text{(as } c \to +\infty, \text{ for } |f|_{\mathfrak{B}^1} \leq 1)$$

giving compactness. ///

2. *Mellin transform functionals in* $\mathfrak{B}^{-1}$

With potential $q(x+iy) \gg y^\alpha$ as $y \to +\infty$, a certain range of Mellin transform maps are in $\mathfrak{B}^{-1}(-\Delta+q)$:

[2.0.1] Theorem: For $\frac{1}{2} \le \mathrm{Re}(s) < \frac{\alpha}{2}$, the *Mellin distribution*

$$\mu_s(f) \;=\; \Lambda(f,s) \;=\; \int_0^\infty y^{s-\frac{1}{2}}\, f(iy)\, \frac{dy}{y} \qquad\qquad (\text{for } f \in C_c^\infty(\Gamma\backslash\mathfrak{H}))$$

is in the Hilbert-space dual $\mathfrak{B}^{-1}(-\Delta+q)$ of $\mathfrak{B}^{+1}(-\Delta+q)$.

Proof: First, an estimate on $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ in terms of its $\mathfrak{B}^{+1}$-norm is obtained from Plancherel applied to the Fourier expansion of $f(x+iy)$ as a periodic function of x:

$$\infty \;>\; |f|^2_{\mathfrak{B}^1} \;=\; \int_{\Gamma\backslash\mathfrak{H}} (-\Delta+q)f\cdot\overline{f}\;\frac{dx\,dy}{y^2} \;\ge\; \int_{y\ge 1}\int_{\mathbb{Z}\backslash\mathbb{R}} (-\Delta+q)f\cdot\overline{f}\;\frac{dx\,dy}{y^2}$$

$$\gg_q \int_{y\ge 1}\int_{\mathbb{Z}\backslash\mathbb{R}} (-y^2\frac{\partial^2}{\partial x^2} + y^\alpha) f\cdot\overline{f}\;\frac{dx\,dy}{y^2} \;\gg\; \int_{y\ge 1}\sum_n (y^2n^2+y^\alpha)\cdot|c_n(y)|^2\;\frac{dy}{y^2}$$

$$\ge\; \int_1^\infty \sum_n \; y^{\alpha-1}\,(n^2+1)\,|c_n(y)|^2\;\frac{dy}{y}$$

Meanwhile, for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ a bound on $\mu_s(f)$ has a similar expression, as follows. By the functional equation $s \leftrightarrow 1-s$, take $\sigma = \mathrm{Re}(s) \ge \frac{1}{2}$. Use $f(-1/z) = f(z)$:

$$|\mu_s(f)| \;=\; \Big|\int_0^\infty y^{s-\frac{1}{2}}\, f(iy)\,\frac{dy}{y}\Big| \;=\; \Big|\int_1^\infty (y^{s-\frac{1}{2}} + y^{\frac{1}{2}-s})\, f(iy)\,\frac{dy}{y}\Big| \;\le\; 2\int_1^\infty y^{\sigma-\frac{1}{2}}\,|f(iy)|\,\frac{dy}{y}$$

For any $\delta > 0$, by Cauchy-Schwarz-Bunyakowsky, and at the end remembering the earlier estimate,

$$|\mu_s(f)| \;\ll\; \int_1^\infty y^{\sigma-\frac{1}{2}}\,|f(iy)|\,\frac{dy}{y} \;\le\; \int_1^\infty \sum_n \; y^{\sigma-\frac{1}{2}}\,|c_n(y)|\;\frac{dy}{y}$$

$$=\; \int_1^\infty \sum_n \; \frac{1}{y^\delta\sqrt{n^2+1}} \;\cdot\; y^{\sigma-\frac{1}{2}+\delta}\sqrt{n^2+1}\,|c_n(y)|\;\;\frac{dy}{y}$$

$$\le\; \Big(\int_1^\infty \sum_n \; \frac{1}{y^{2\delta}(n^2+1)}\;\frac{dy}{y}\Big)^{\frac{1}{2}} \;\cdot\; \Big(\int_1^\infty \sum_n \; y^{2\sigma-1+2\delta}(n^2+1)\,|c_n(y)|^2\;\;\frac{dy}{y}\Big)^{\frac{1}{2}}$$

$$\ll_\delta \;\Big(\int_1^\infty \sum_n \; y^{2\sigma-1+2\delta}(n^2+1)\,|c_n(y)|^2\;\;\frac{dy}{y}\Big)^{\frac{1}{2}} \;\ll_\delta\; |f|_{\mathfrak{B}^1} \qquad\qquad (\text{for } 2\sigma-1+2\delta \le \alpha-1)$$

When $\sigma < \frac{\alpha}{2}$, the condition $2\sigma-1+2\delta \le \alpha-1$ holds for some $\delta > 0$. The estimate on $\mu_s(f)$ holds for $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ and then by continuity for $f \in \mathfrak{B}^{+1}$. ///

3. *Hilbert-Schmidt resolvent of* $-\Delta + q$ *for* $q \gg y^2$

When $-\Delta + q$ has Hilbert-Schmidt resolvent, *all* transition maps $\mathfrak{B}^n(-\Delta + q) \to \mathfrak{B}^{n-2}(-\Delta + q)$ in the projective limit are Hilbert-Schmidt: for orthonormal basis $\{u_i\}$ of eigenfunctions for $L^2(\Gamma\backslash\mathfrak{H})$, with eigenvalues $\lambda_i > 0$, the vectors $u_i/\lambda_i^{n/2}$ form an orthonormal basis for $\mathfrak{B}^n = \mathfrak{B}^n(-\Delta + q)$. With respect to these orthonormal bases, the inclusions are simply multiplication maps

$$\sum_i c_i \frac{u_i}{\lambda_i^{\frac{n}{2}}} \longrightarrow \sum_i c_i \cdot \lambda_i^{-1} \frac{u_i}{\lambda_i^{\frac{n-2}{2}}}$$

Such a map is Hilbert-Schmidt if and only if

$$\sum_i (\lambda_i^{-1})^2 < \infty$$

The resolvent of the Friedrichs extension of $-\Delta + q$ has eigenvalues λ_i^{-1}, and the Hilbert-Schmidt property is the same inequality. In this situation $\mathfrak{B}^{+\infty} = \lim_n \mathfrak{B}^n$ is nuclear Fréchet, giving a Schwartz kernel theorem.

[3.0.1] Theorem: $-\Delta + q$ has Hilbert-Schmidt resolvent for $q(x+iy) \gg y^2$ as $y \to +\infty$.

Proof: As in the proof of compactness of the resolvent, the fact that $H^s(\mathbb{T}^2) \to H^{s-2}(\mathbb{T}^2)$ is Hilbert-Schmidt reduces discussion to consideration of the geometrically simpler non-compact part of $\Gamma\backslash\mathfrak{H}$. Specifically, it suffices to consider the restriction S of $-\Delta + q$ to test functions on the tapering cylinder $X = \mathbb{T}^1 \times [1, \infty)$, with measure $\frac{dx\,dy}{y^2}$, and to take $q(x+iy) = y^2$.

Thus, the domain of S includes test functions on X vanishing to infinite order on the boundary $\partial X = \mathbb{T}^1 \times \{1\}$. Let $\widetilde{S}$ be the Friedrichs self-adjoint extension of S.

On this non-compact but geometrically simpler fragment of $\Gamma\backslash\mathfrak{H}$, the circle group $\mathbb{T}$ acts, and commutes with S and $\widetilde{S}$. Thus, $L^2(X)$ decomposes orthogonally into components indexed by characters $\psi_n(x) = e^{inx}$ of $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. On the n^{th} component, the differential equation for that component of a fundamental solution u_a at a is

$$\delta_a = (-\Delta + q)\Big(e^{inx}\, u_a(y)\Big) = y^2(n^2 - \frac{\partial^2}{\partial y^2} + 1)u_a(y) = y^2\Big(- u_a'' + (n^2+1)u_a\Big)$$

simplifying, conveniently, to a constant-coefficient equation

$$\frac{1}{a^2}\,\delta_a = -u_a'' + (n^2+1)u_a \qquad \text{(with } a > 1)$$

We can follow the usual prescription for piecing together u_a from solutions $e^{\pm cy}$ to the corresponding homogeneous equation $-u'' + (n^2+1)u = 0$, letting $c = \sqrt{n^2+1} \ge 1$. That is, $u_a(y)$ must have moderate-enough growth as $y \to +\infty$ so that it is in $L^2(X)$ with measure $dx\,dy/y^2$, and go to zero as $y \to 1^+$, in addition to being continuous but non-smooth enough at $y = a$ to produce the required multiple of δ_a. Thus, u_a must be of the form

$$u_a(y) = \begin{cases} A_a e^{cy} + B_a e^{-cy} & \text{(for } 1 < y < a) \\ C_a e^{-cy} & \text{(for } a < y) \end{cases}$$

for some constants A_a, B_a, C_a, since e^{cy} grows too rapidly as $y \to +\infty$. The conditions are

$$\begin{cases} A_a e^{c} + B_a e^{-c} = 0 & \text{(vanishing at } y \to 1^+) \\ A_a e^{ca} + B_a e^{-ca} = C_a e^{-ca} & \text{(continuity at } y = a) \\ -c\,C_a e^{-ca} - (cA_a e^{ca} - cB_a e^{-ca}) = \frac{1}{a^2} & \text{(change of slope by } \frac{1}{a^2} \text{ at } y = a) \end{cases}$$

From the first equation, $B_a = -e^{2c} \cdot A_a$, and the system becomes

$$\begin{cases} A_a(e^{ca} - e^{2c}e^{-ca}) = C_a e^{-ca} & \text{(continuity at } y = a) \\ -c\,C_a e^{-ca} - cA_a\Big(e^{ca} + e^{2c}e^{-ca}\Big) = \frac{1}{a^2} & \text{(change of slope by } \frac{1}{a^2} \text{ at } y = a) \end{cases}$$

Substituting $C_a = A_a \cdot (e^{2ca} - e^{2c})$, from the first equation, into the second, gives

$$A_a \cdot \Big(- c(e^{2ca} - e^{2c})e^{-ca} - c(e^{ca} + e^{2c}e^{-ca}\Big) \;=\; \frac{1}{a^2}$$

simplifying to $A_a = -e^{-ca}/2ca^2$. Then

$$C_a \;=\; A_a \cdot (e^{2ca} - e^{2c}) \;=\; \frac{e^{2c}e^{-ca} - e^{ca}}{2ca^2}$$

so

$$u_a(y) \;=\; C_a \cdot e^{-cy} \;=\; \frac{e^{2c}e^{-ca} - e^{ca}}{2ca^2} \cdot e^{-cy} \qquad \text{(for } y > a)$$

Since $(c^2 - \frac{\partial^2}{\partial y^2})u = f$ is solved by

$$u(y) \;=\; \int_1^\infty a^2 \cdot u_a(y)\, f(a)\, da$$

the symmetry of $-\Delta + q$ with respect to the measure dy/y^2 implies that $a^2 \cdot u_a(y)$ is symmetric in y, a, and the integral kernel for the resolvent is

$$a^2 \cdot u_a(y) \;=\; \begin{cases} \dfrac{e^{2c}e^{-ca} - e^{ca}}{2c} \cdot e^{-cy} & \text{(for } y > a) \\[2ex] \dfrac{e^{2c}e^{-cy} - e^{cy}}{2c} \cdot e^{-ca} & \text{(for } 1 < y < a) \end{cases}$$

The resolvent being Hilbert-Schmidt is equivalent to

$$\int_1^\infty \int_1^\infty |a^2 \cdot u_a(y)|^2 \, \frac{da}{a^2}\, \frac{dy}{y^2} \;<\; \infty$$

By symmetry, it suffices to integrate over $1 < a < y < \infty$, and

$$\int\int_{1<a<y} |a^2 \cdot u_a(y)|^2 \, \frac{da}{a^2}\, \frac{dy}{y^2} \;=\; \int\int_{1<a<y} \frac{|e^{4c}e^{-2ca} - 2e^{2c} + e^{2ca}| \cdot e^{-2cy}}{4c^2} \, \frac{da}{a^2}\, \frac{dy}{y^2}$$

$$\ll\; \frac{1}{n^2+1} \int\int_{1<a<y} (e^{4c}e^{-2ca} + 2e^{2c} + e^{2ca}) \cdot e^{-2cy} \, \frac{da}{a^2}\, \frac{dy}{y^2}$$

Replacing y, a by $y+1, a+1$, the integral becomes

$$\int\int_{0<a<y} (e^{-2ca} + 2 + e^{2ca}) \cdot e^{-2cy} \, \frac{da}{(a+1)^2}\, \frac{dy}{(y+1)^2} \;\ll\; \int\int_{0<a<y} \frac{da}{(a+1)^2}\, \frac{dy}{(y+1)^2}$$

$$\ll\; \int_0^\infty \frac{da}{(a+1)^2} \cdot \int_0^\infty \frac{dy}{(y+1)^2} \;<\; \infty$$

Thus, the n^{th} component of the integral kernel has L^2 norm bounded by a uniform constant multiple of $1/(n^2+1)$. The sum over $n \in \mathbb{Z}$ is finite, proving that the resolvent is Hilbert-Schmidt. ///

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