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Standard estimates for $SL_2(\mathbb{Z}[i])\backslash SL_2(\mathbb{C})/SU(2)$

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This is a simple example of a general phenomenon. Examples in $SL_2(\mathbb{R})$ ares carried out in [Iwaniec 2002]. For $\Gamma = SL_2(\mathbb{Z}[i])$, $G = SL_2(\mathbb{C})$, and $K = SU(2)$, we prove the *standard estimate*

$$\sum_{|s_F|\le T} |F(g)|^2 \;+\; \frac{1}{2\pi}\int_{-T}^{T} |E_{\frac{1}{2}+it}(g)|^2\,dt \quad \ll_C \quad T^3 \qquad\qquad \text{(uniformly for } g \text{ in a compact } C\subset G)$$

for cuspforms F with eigenvalues

$$\lambda_F \;=\; s_F(s_F-1)$$

for the Laplacian $\mathcal{D}$, and Eisenstein series E_s. We normalize the dependence of E_s on the parameter s so that the functional equation relates E_s and E_{1-s}.

As usual, we consider *integral* operators attached to compactly supported measures η on the group G, and exploit the intrinsic sense of such operators on any reasonable representation space for G, for example, Hilbert, Banach, Fréchet, and LF (strict colimits of Fréchet), or, generally,, quasi-complete, locally convex spaces. For a representation π, V of G, and a compactly-supported measure η, the action is

$$\eta\cdot v \;=\; \int_G \pi(g)(v)\,d\eta(g) \qquad\qquad \text{(for } v\in V)$$

The general theory of Gelfand-Pettis integrals assures the reasonable behavior of such integrals.

The non-trivial but memorable fact used in the proof, illustrated in the case of $G = SL_2(\mathbb{C})$, is that a waveform f, an eigenfunction for the G-invariant Laplacian $\mathcal{D}$ in $L^2(\Gamma\backslash G/K)$, generates an irreducible representation of G under right translation, specifically, an *unramified principal series* I_s. [1] The same is true of Eisenstein series E_s more immediately. We index the character defining the unramified principal series I_s so that the standard intertwining operators go from I_s to I_{1-s}.

Thus, a waveform f (or Eisenstein series E_s) is the unique spherical vector in the copy of the unramified principal series representation it generates, up to a constant. Thus, for any left-and-right K-invariant compactly-supported measure η the integral operator action

$$(\eta\cdot f)(x) \;=\; \int_G \pi(y)f(xy)\,d\eta(y)$$

produces another right K-invariant vector in the representation space of f. Necessarily $\eta\cdot f$ is a scalar multiple of f. Let $\chi_f(\eta)$ denote the eigenvalue:

$$\eta\cdot f \;=\; \chi_f(\eta)\cdot f \qquad\qquad \text{(with } \chi_f(\eta)\in\mathbb{C})$$

This is an intrinsic representation-theoretic relation, so the scalar $\chi_f(\eta)$ can be computed in any model of the representation. We choose an unramified principal series

$$I_s \;=\; \Big\{\text{smooth } K\text{-finite } \varphi \;:\; \varphi(\begin{pmatrix} a & * \\ 0 & d\end{pmatrix}\cdot g) \;=\; \Big|\frac{a}{d}\Big|^{2s}\cdot\varphi(g)\Big\} \qquad\qquad \text{(with } s\in\mathbb{C})$$

[1] The general theory of spherical functions shows that, generally, eigenfunctions for all left G-invariant differential operators on G/K generate principal series. Often, the center of the enveloping algebra *surjects* to that collection of differential operators: for classical groups this holds. However, [Helgason 1984] gives examples of non-surjection among exceptional groups.

On I_s, the Laplacian or Casimir has eigenvalue $\lambda_f = s(s-1)$.

[1.1] Choice of integral operator Let $\|g\|$ be the square of the *operator norm* on G for the standard representation of G on $\mathbb{C}^2$ by matrix multiplication. In a Cartan decomposition,

$$\|k_1 \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \cdot k_2\| \;=\; e^r \qquad\qquad (\text{with } k_1, k_2 \in K,\, r \geq 0)$$

This norm gives a left G-invariant metric $d(,)$ on G/K by

$$d(gK,\, hK) \;=\; \log \|g^{-1}h\| \;=\; \log \|h^{-1}g\|$$

The triangle inequality follows from the submultiplicativity of the norm.

Take η to be the characteristic function of the left and right K-invariant set of group elements of norm at most e^δ, with small $\delta > 0$. That is,

$$\eta(g) \;=\; \begin{cases} 1 & (\text{for } \|g\| \leq e^\delta) \\ 0 & (\text{for } \|g\| > e^\delta) \end{cases}$$

or

$$\eta\Big(k_1 \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \cdot k_2\Big) \;=\; \begin{cases} 1 & (\text{for } r \leq \delta) \\ 0 & (\text{for } r > \delta) \end{cases} \qquad\qquad (\text{with } r \geq 0)$$

[1.2] Upper bound on a kernel The map $f \to (\eta \cdot f)(x)$ on automorphic forms f can be expressed as integration of f against a sort of automorphic form q_x by winding up the integral, as follows.

$$(\eta \cdot f)(x) \;=\; \int_G f(xy)\,\eta(y)\,dy \;=\; \int_G f(y)\,\eta(x^{-1}y)\,dy \;=\; \int_{\Gamma\backslash G} \Big(\sum_{\gamma\in\Gamma} f(\gamma y)\,\eta(x^{-1}\gamma y)\Big)\,dy$$

$$=\; \int_{\Gamma\backslash G} f(y) \cdot \Big(\sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)\Big)\,dy$$

Thus, for $x, y \in G$ put

$$q_x(y) \;=\; \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)$$

The norm-squared of q_x, as a function of y alone, is

$$|q_x|^2_{L^2(\Gamma\backslash G)} \;=\; \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} \eta(x^{-1}\gamma\gamma y)\overline{\eta}(x^{-1}\gamma' y)\,dy \;=\; \int_G \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)\overline{\eta}(x^{-1}y)\,dy$$

after unwinding. For both $\eta(x^{-1}\gamma y)$ and $\eta(x^{-1}y)$ to be non-zero, the distance from x to both y and γy must be at most δ. By the triangle inequality, the distance from y to γy must be at most 2δ. For x in a fixed compact C, this requires that y be in ball of radius δ, and that $\gamma y = y$. Since K is compact and Γ is discrete, the isotropy groups of all points in G/K are finite. Thus,

$$|q_x|^2_{L^2(\Gamma\backslash G)} \;\ll\; \int_{d(x,y)\leq\delta} 1\,dy \;\asymp\; \delta^3$$

[1.3] Lower bound on eigenvalues A non-trivial *lower* bound for $\chi_f(\eta)$ can be given for $\delta \ll 1/t_f$, as follows. With spherical function φ^o in the s^{th} principal series, the corresponding eigenvalue is

$$\chi_s(\eta) \;=\; \int_G \eta(g)\,\varphi^o(g)\,dg \;=\; \int_{r\leq\delta} \varphi^o(k \cdot \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix})\,dg$$

In fact, a qualitative argument clearly indicates the outcome, although we will also carry out a more explicit computation. For the qualitative argument, we need qualitative metrical properties of the Iwasawa decomposition. Let P^+ be the upper-triangular matrices in G with positive real entries, and $K = SU(2)$. Let $g \to p_g k_g$ be the decomposition. We claim that $\|g\| \le \delta$ implies $\|p_g\| \ll \delta$ for small $\delta > 0$. This is immediate, since the Jacobian of the map $P^+ \to G/K$ near $e \in P^+$ is *invertible*.

But, also, the Iwasaw decomposition is easily computed here, and the integral expressing the eigenvalue can be estimated explicitly: elements of K can be parametrized as

$$k = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \qquad \text{(where } |\alpha|^2 + |\beta|^2 = 1\text{)}$$

and let $a = e^{r/2}$. Then

$$k \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} * & * \\ -a\beta & \alpha/a \end{pmatrix}$$

Right multiplication by a suitable element k_2 of $SU(2)$ rotates the bottom row to put the matrix into P^+:

$$k \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot k_2 = \begin{pmatrix} * & * \\ 0 & \sqrt{(-a|\beta|)^2 + (|\alpha|/a)^2} \end{pmatrix}$$

Thus,

$$\chi_s(\eta) = \int_{r \le \delta} \Big((-a|\beta|)^2 + (|\alpha|/a)^2\Big)^{-s} \, dg$$

Rather than compute the integral exactly, make δ small enough to give a lower bound on the integrand, such as would arise from

$$\Big|\Big((-a|\beta|)^2 + (|\alpha|/a)^2\Big)^{-s} - 1\Big| < \tfrac{1}{2} \qquad \text{(for all elements of } K\text{)}$$

Since $|\alpha|^2 + |\beta|^2 = 1$, for small r,

$$(-e^{r/2}|\beta|)^2 + (|\alpha|/e^{r/2})^2 = e^r|\beta|^2 + |\alpha|^2/e^r \asymp (1+r)|\beta|^2 + (1-r)|\alpha|^2 \ll 1 + r$$

Thus, for small $0 \le r \le \delta$,

$$\big|\big(e^r|\beta|^2 + |\alpha|^2/e^r\big)^{-s} - 1\big| \ll |s| \cdot r$$

Thus, $0 \le r \le \delta \ll \frac{1}{|s|}$ suffices to make this less than $\frac{1}{2}$. That is, with η the characteristic function of the δ-ball, we have the lower bound

$$|\chi_s(\eta)| = \int_G \eta(g)\, \varphi^o(g)\, dg \gg \int_{r \le \delta} 1 = \mathrm{vol}\,(\delta\text{-ball}) \asymp \delta^3 \qquad (\eta \text{ char fcn of } \delta\text{-ball, for } |s| \ll 1/\delta,\ )$$

Taking δ as large as possible compatible with $\delta \ll 1/|s|$ gives the bound

$$\chi_s(\eta) \gg \delta^3 \qquad (\text{for } |s| \ll 1/\delta,\ \eta \text{ the characteristic function of } \delta\text{-ball})$$

From the L^2 automorphic spectral expansion of q_x, apply Plancherel

$$\sum_F |\langle q_x, F\rangle|^2 + \frac{|\langle q_x, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\langle q_x, E_s\rangle|^2 \, dt = |q_x|^2_{L^2(\Gamma\backslash G/K)} \ll \delta^3$$

Truncating this to Bessel's inequality and dropping the single residual term,

$$\sum_{|s_F| \le T} |\langle q_x, F\rangle|^2 + \frac{1}{2\pi} \int_{-T}^{+T} |\langle q_x, E_s\rangle|^2 \, dt \ll \delta^3$$

Recall that for a the spherical vector $f \in I_s$

$$\langle q_x, f\rangle \;=\; \chi_s(\eta)\cdot f$$

and use the inequality $\chi_s(\eta) \gg \delta^3$ from above for this restricted parameter range, obtaining

$$\sum_{|s_F|\le T} \big(\delta^3\cdot|F(x)|\big)^2 + \int_{-T}^{+T} \big(\delta^3\cdot|E_s(x)|\big)^2\,dt \;\ll\; \delta^3$$

Multiply through by $T^6 \asymp 1/\delta^6$ to obtain the **standard estimate**

$$\sum_{|s_F|\le T} |F(x)|^2 + \int_{-T}^{+T} |E_s(x)|^2\,dt \;\ll\; T^3$$

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