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Representations of $GL(2)$ and $SL(2)$ over finite fields

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Irreducible complex representations of $GL(2)$ and $SL(2)$ over a finite field can be classified by methods useful for p-adic reductive groups and real Lie groups. (See Piatetski-Shapiro's inspiring A.M.S. Memoir on this subject.) That is, we first see that most **principal series representations** are *irreducible.* We determine the irreducible constituents of the *irregular* principal series. We prove the uniqueness of Whittaker models by showing that the endomorphism ring of the space of Whittaker functions is commutative.

We resist using techniques special to finite groups and finite-dimensional representations, to practice more sophisticated techniques. For example, we minimize invocation of *dimension* or *cardinality*, as well as the theorem (recalled below) on *unitarizability* and *complete reducibility.* Still, special explicit facts do help gauge the effectiveness of the methods here.

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1. *Background*

The simplicity of the statements and/or proofs of the following for finite-dimensional complex representations of finite groups is convenient in the short term, but gives a false impression about reasonable goals in other important situations.

As usual, a **representation** of a group G on a complex vector space V is a group homomorphism

$$\pi \;:\; G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

to complex-linear automorphisms of V. One style of notation is to say that the *ordered pair* (π, V) is the representation, and to write the action of an element $g \in G$ on $v \in V$ as

$$g \times v \;\longrightarrow\; \pi(g)(v)$$

This notation is burdensome, and usually unnecessary. Instead, we will typically write

$$g \times v \;\longrightarrow\; g \cdot v$$

[1] When using the *pi*-less and parenthesis-less notation for a representation (π, V), we may often avoid naming the homomorphism π, and refer to *the representation* V, freeing up symbols for other uses. [2]

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[1] The *pi*-less and parenthesis-less style of notation is compatible with the standard notation for modules over rings. The decrease in visual noise is another strong recommendation.

[2] There is also a notational style in which the representation space for a homomorphism π is invariably denoted V_π. A more serious breach of notational propriety is a common abuse of notation, in which, for a representation properly denoted (π, V), instead of writing $v \in V$ we write $v \in \pi$. That is, the symbol for the group homomorphism is also used as a symbol for the vector space. Apart from the confustion this convention may cause, it has the virtue of saving a symbol for the name of the vector space. Indeed, there is no need for confustion, since the notation $v \in \pi$ has no immediate alternative meaning.

All representations will be on *finite-dimensional complex* vector spaces.

Recall that a representation V of a group G is *irreducible* if it has no proper subrepresentations, that is, no subrepresentations other than the trivial representation $\{0\}$ and the whole representation V itself.

[1.0.1] **Theorem:** *(Unitarizability)* A finite-dimensional complex representation V of a finite[3] group G has a G-invariant[4] hermitian inner product $\langle , \rangle$. ///

[1.0.2] **Corollary:** *(Complete Reducibility)* For a G-subrepresentation W of a finite-dimensional complex representation V of a finite group G, there is another G-subrepresentation U such that

$$V = U \oplus W$$

With a G-invariant inner product on V this direct sum decomposition is *orthogonal*. ///

[1.0.3] **Theorem:** *(Schur's Lemma)* For an irreducible V of G,

$$\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot 1_V$$

That is, the endomorphisms of V commuting with G are only the scalars. ///

[1.0.4] **Corollary:** *(Of previous corollary and last theorem: Endomorphism algebra criterion for irreducibility)* A G-representation V is irreducible if and only if $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V) = 1$. ///

[1.0.5] **Theorem:** Summing over isomorphism classes for the irreducibles of a finite group G,

$$\sum_V (\dim V)^2 = \text{ order of } G$$

(From the decomposition of the biregular representation. The form of this decomposition holds for compact groups.) ///

The following helpful result for finite groups has no simple general counterpart:

[1.0.6] **Theorem:** The set of (isomorphism classes of) irreducible complex representations of a finite group G is of the same cardinality as the collection of conjugacy classes in G. ///

Last, finite-dimensional representations V are **reflexive** in the usual sense that the second dual V^{**} is naturally isomorphic to V. (The same conclusion holds for *admissible* smooth representations of p-adic and Lie groups.)

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2. *Principal series representations of* $GL(2)$

Let k be a finite field with q elements. Let $G = GL(2,k)$ or $G = SL(2,k)$. Let

$$P = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\} \quad N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G\} \quad M = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in G\} \quad w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

[3] This result and its immediate corollaries extend to *compact* topological groups, using Haar measure in place of counting measure.

[4] The G-invariance has the natural meaning, that $\langle gx, gy \rangle = \langle x, y \rangle$ for all $g \in G$, $x, y \in V$.

The subgroup P is the *standard parabolic* subgroup, N its *unipotent radical*, and M the standard *Levi component* of P. The subgroup P is the semidirect product of M and N, with M normalizing N. This w_o is the *longest Weyl element*.

The important tangible family of representations of G is the **principal series** of representations I_χ of G attached to characters (meaning one-dimensional representations)

$$\chi : M \longrightarrow \mathbb{C}^\times$$

For $G = SL(2)$ we have $M \approx k^\times$ so these characters are characters χ_1 of $k^\times$ via

$$\chi\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \chi_1(a)$$

For $G = GL(2)$ we have $M \approx k^\times \times k^\times$ and these characters are pairs (χ_1, χ_2) of characters of $k^\times$ via

$$\chi\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \chi_1(a)\chi_2(d)$$

In either case, extend χ to P by being identically 1 on N. Then the χ^{th} principal series representation of G attached to χ is [5] to be the $\mathbb{C}$-vectorspace of functions

$$I_\chi = \mathrm{Ind}_P^G\,\chi = \{ \ \mathbb{C}\text{-valued functions } f \text{ on } G \ : f(pg) = \chi(p)\, f(g) \ \text{ for all } p \in P,\, g \in G \ \}$$

The *action* of G on $\mathrm{Ind}_P^G\chi$ is by the *right regular* representation

$$R_g(f)(x) = f(xg)$$

An important aspect of representations of G induced from subgroups is that they are *constructed*, so *exist*. One would hope to construct many (if not all) *irreducibles* by inducing. As below, principal series representations with $\chi(wmw^{-1}) \neq \chi(m)$ (for $m \in M$) are irreducible, and these irreducibles are about half of all irreducibles of G.

Induced representations have a computationally convenient feature, namely, [6]

[2.0.1] Theorem: *(Frobenius Reciprocity)* For a representation σ of a subgroup H of G, and for a representation V of G, there is a natural isomorphism

$$F : \mathrm{Hom}_G(V, \mathrm{Ind}_H^G\,\sigma) \approx \mathrm{Hom}_H(\mathrm{Res}_H^G\, V, \sigma)$$

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[5] This is an *induced representation*, more generally defined as follows for finite groups. For a representation σ of a subgroup K of a group H,

$$\mathrm{Ind}_K^H\,\sigma = \{ \ \sigma\text{-valued functions } f \text{ on } H \ : f(kh) = \chi(k)\, f(h) \ \text{ for all } k \in K,\, h \in H \ \}$$

This simplifies when σ is one-dimensional, as with I_χ, by identifying the representation space with $\mathbb{C}$, yielding complex-valued functions rather than representation-space-valued functions on G. The *action* of H on $\mathrm{Ind}_K^H\sigma$ is by the *right regular* representation

$$R_h(f)(x) = f(xh)$$

[6] In the long run, it is better to *characterize* the induced representation as an object making Frobenius Reciprocity hold, rather than *constructing* a representation and then proving that it has the property. Frobenius Reciprocity is an instance of an *adjunction relation* for *adjoint functors*.

of $\mathbb{C}$-vectorspaces, where Res_H^G is the forgetful functor which considers G-representations as mere H-representations. For $v \in V$, the isomorphism F is

$$F(\Phi)(v) \;=\; \Phi(v)(1_G)$$

and its inverse is

$$\left(F^{-1}(\varphi)(v)\right)(g) \;=\; R_g\left(\varphi(v)\right)$$

(Given the formulas, the proof is straightforward.) ///

For a *one-dimensional* irreducible $\sigma : H \to \mathbb{C}^\times$ of a group H, a **σ-isotypic** representation V of H is a (possibly large) representation V of H on which H acts entirely by σ, in the sense that [7]

$$h \cdot v \;=\; \sigma(h) \cdot v \qquad\qquad (\text{for all } v \in V,\, h \in H)$$

For a representation V of H, the *σ-isotype* V^σ of V is the smallest subrepresentation $i : V^\sigma \to V$ of V such that any H-morphism

$$\varphi \;:\; W \;\longrightarrow\; V$$

of a σ-isotypic H-representation W uniquely factors through V^σ, namely there is a unique $\varphi_o : W \to V^\sigma$ such that

$$\varphi \;=\; i \circ \varphi_o \;:\; W \;\longrightarrow\; V^\sigma \to V$$

Existence of the isotype is proven by the *construction*:

$$V^\sigma \;=\; \sum_{\varphi:\sigma\to V} \mathrm{Im}\,\varphi$$

Proof that the construction meets the defining characterization is an exercise.

The **σ-co-isotype** V_σ of a representation V of H is the smallest H-space quotient of V such that any H-homomorphism $\varphi : V \to W$ with W σ-isotypic factors through V_σ. A *construction* is

$$V_\sigma \;=\; V/ \bigcap_{\varphi:V\to\sigma} \ker\varphi$$

Proof that the construction meets the defining characterization is an exercise.

[2.0.2] **Remark:** In situations where complete reducibility holds, co-isotypes are subrepresentations, thus are isotypes. Nevertheless, it is useful to make clear this distinction, since co-isotypes are not isotypes *generally.*

Consider a representation V of G as a representation of the subgroup N. Our convention is that **the trivial representation** of N is a one-dimensional vector space on which N acts trivially. Changing the notation slightly from the previous paragraph, the **Jacquet module** $J_N V$ of V is defined to be

$$\text{Jacquet module } J_N V \text{ of } V \;=\; \text{co-isotype for trivial representation of } N \;=\; V_N$$

The *Jacquet functor* J_N is

$$J_N \;:\; V \;\longrightarrow\; V_N$$

It is an exercise that, since M normalizes N, $J_N V = V_N$ is a representation of M. Thus, the Jacquet functor J_N is a functor from G-representations to M-representations.

[7] For irreducibles of dimension greater than 1 the notion of an isotypic representation is more delicate, varying more with the context.

We will generally suppress the explicit notation Res^G_H for a forgetful restriction functor, as its application will be clear from context.

Combining the defining property of the Jacquet module with Frobenius Reciprocity,

[2.0.3] **Corollary:** For representations V of G there is a $\mathbb{C}$-linear isomorphism

$$\mathrm{Hom}_G(V, I_\chi) \approx \mathrm{Hom}_M(V_N, \chi)$$

(Proof is an exercise.)

[2.0.4] **Corollary:** An irreducible representation V of G with $V_N \neq \{0\}$ is isomorphic to a subrepresentation of a principal series I_χ for some χ. On the other hand, for $V_N = \{0\}$, (non-zero) V is *not* isomorphic to a subrepresentation of *any* principal series.

Proof: The representation space V_N of the group M is finite-dimensional, hence has an irreducible quotient $\varphi : V_N \to \chi$. (Exercise.) In particular, this map φ is not 0. Since M is abelian χ is one-dimensional. (Exercise.) Thus, via the inverse L^{-1} of the isomorphism

$$L : \mathrm{Hom}_G(V, I_\chi) \approx \mathrm{Hom}_M(V_N, \chi)$$

we obtain a non-zero $T^{-1}\varphi \in \mathrm{Hom}_G(V, I_\chi)$. ///

A representation V is **supercuspidal** when $V_N = \{0\}$. Thus, by this definition and by the above corollary, the irreducibles of G *either* imbed into a principal series *or* are supercuspidal. (For larger groups such as $GL(3,k)$ and $SL(3,k)$ there is a larger array of intermediate cases.)

The next issue is assessment of the irreducibility of the principal series I_χ, proving below that I_χ is irreducible if χ is *regular*, meaning that $\chi^w \neq \chi$, where for $m \in M$ the character χ^w is

$$\chi^w(m) = \chi(wmw^{-1})$$

We continue to exploit the connection between imbeddability into principal series and non-vanishing of the Jacquet functor. The first observation is that *if* V were a *proper* subrepresentation of a principal series representation I_χ, then the quotient I_χ/V would be non-zero, and thus would have an irreducible quotient π. We want to show that $\pi_N \neq 0$, so that (from above) π is a subrepresentation of some principal series I_β, giving a non-zero G-intertwining $I_\chi \to I_\beta$.

[2.0.5] **Proposition:** For a G-representation V, the kernel of the Jacquet map $J_N : V \to V_N$ is generated by all expressions

$$v - n \cdot v$$

for $v \in V$ and $n \in N$. Also,

$$\ker J_N = \{v \in V : \int_N n \cdot v \, dn = 0\}$$

(where we have written an integral even though the group N is finite.)

Proof: Under *any* N-map $r : V \to W$ with N acting trivially on W,

$$r(v - nv) = rv - r(nv) = rv - n(rv) = rv - rv = 0$$

so the elements $v - nv$ are in the kernel of the quotient map to the Jacquet module. On the other hand, the linear span of these elements is stable under N, so we may form the quotient of V by these elements. This proves that the first description of the kernel is correct.

To prove the second characterization, suppose that

$$\int_N n\cdot v\,dn \;=\; 0$$

Then

$$v \;=\; v-0 \;=\; v-\frac{1}{\text{meas}\,(N)}\int_N n\cdot v\,dn \;=\; \frac{1}{\text{meas}\,(N)}\int_N (v-n\cdot v)\;dn$$

a finite sum, expressing v as a linear combination of the desired form. On the other hand,

$$\int_N n\cdot(v-n_o\cdot v)\;dn \;=\; \int_N n\cdot v\,dn-\int_N (nn_o\cdot v\,dn \;=\; \int_N n\cdot v\,dn-\int_N n\cdot v\,dn \;=\; 0$$

by changing variables in the second integral. ///

[2.0.6] **Theorem:** The Jacquet functor $J_N : V\to V_N$ is an *exact functor*, meaning that for $f:A\to B$ and $g:B\to C$ are maps such that

$$0\to A\to B\to C\to 0$$

is a short exact sequence of G-representations, the induced maps on Jacquet modules give an exact sequence

$$0\to A_N\to B_N\to C_N\to 0$$

[2.0.7] **Remark:** This theorem can be interpreted as asserting that the group homology of N is always trivial (above degree 0), in the following sense. Even in a somewhat larger context, it is true that co-isotype functors are *right* exact and isotype functors are *left* exact, for reasons noted in the proof below. Given a projective resolution

$$\ldots\xrightarrow{d}F^2\xrightarrow{d}F^1\xrightarrow{d}F^0\to V\to 0$$

of an N-representation V (by N-representations F^i) the *group homology* of V the homology

$$H_n(V) \;=\; \frac{\ker d \text{ on } F^n_N}{d(F^{n+1}_N)}$$

of the sequence

$$\ldots\xrightarrow{d}F^2_N\xrightarrow{d}F^1_N\xrightarrow{d}F^0_N\to 0$$

where, in particular, $H_0(V)=V_N$. That is, the higher group homology modules are the *left derived functors* of the (trivial representation) co-isotype functor. The long exact sequence attached to a short exact sequence

$$0\to A\to B\to C\to 0$$

is

$$\ldots\to H_2(A)\to H_2(B)\to H_2(C)\xrightarrow{\delta}H_1(A)\to H_1(B)\to H_1(C)\xrightarrow{\delta}H_0(A)\to H_0(B)\to H_0(C)\to 0$$

From this and $H_0(V)=V_N$ we have the universal result that

$$H_1(B)\to H_1(C)\xrightarrow{\delta}A_N\;\longrightarrow\;B_N\to C_N\to 0$$

is exact. Similar remarks apply to isotypes and cohomology.

Proof: The right half-exactness is a general property of co-isotypes. [8] That is, the surjectivity of $g : B_N \to C_N$ follows from that of $q \circ g : B \to C_N$ by a very general mechanism. Likewise, since the composite $g \circ f : A \to C$ is 0, certainly

$$q \circ g \circ f : A \to C_N$$

is 0, so the composite $A_N \to B_N \to C_N$ is 0.

The injectivity of $A_N \to B_N$ and the fact that the image of A_N in B_N is the whole kernel of $B_N \to C_N$ are less general, using here the finiteness of the group N. Let $a \in A$ such that $q(fa) = 0 \in B_N$. Then

$$\int_N n \cdot fa\, dn \;=\; 0$$

Since f commutes with the action of N, this gives

$$f\left(\int_N n \cdot a\, dn\right) \;=\; 0$$

By the injectivity of f

$$\int_N n \cdot a\, dn \;=\; 0$$

so $qa = 0 \in A_N$. This proves exactness at the left joint.

When $g(qb) = 0$, $q(gb) = 0$, so

$$\int_N n \cdot gb\, dn \;=\; 0$$

and then the N-homomorphism property of g, namely $ng = gn$, gives

$$g\left(\int_N n \cdot b\, dn\right) \;=\; 0$$

Thus, the integral is in the kernel of g, so is in the image of f. Let $a \in A$ be such that

$$fa \;=\; \int_N n \cdot b\, dn$$

Without loss of generality, $\mathrm{meas}\,(N) = 1$. Then

$$\int_N n' \cdot fa\, dn' \;=\; \int_N \int_N n'n \cdot b\, dn\, dn' \;=\; \int_N \int_N n \cdot b\, dn\, dn'$$

by replacing n by $n'^{-1}n$. This gives

$$\int_N n \cdot (fa - b)\, dn \;=\; 0$$

Thus, $q(fa - b) = 0$ and $f(qa) = qb$. This finishes the proof of exactness at the middle joint. ///

As usual, the $\mathbb{C}$-linear *dual* or *contragredient* representation V^* of a G-representation V is the dual $\mathbb{C}$-vectorspace with the action

$$(g \cdot \lambda)(v) \;=\; \lambda(g^{-1} \cdot v)$$

for $\lambda \in V^*$, $v \in V$, and $g \in G$. Being more careful, since there are two different representations involved, let (π, V) be the given representation, and (π^*, V^*) the dual. The definition of π^* is

$$(\pi^*(g)(\lambda))\,(v) \;=\; \lambda\left(\pi(g^{-1})(v)\right)$$

[8] The fact that the right-exactness instantiates a general property of co-isotypes does not mean that the proof is trivial. The *left*-exactness of *isotypes* $V \to V^\sigma$ is easier to prove.

The mapping $v \times \lambda \to \lambda(v)$ will also be denoted by

$$v \times \lambda \to \lambda(v) \;=\; \langle v, \lambda \rangle$$

[2.0.8] **Proposition:** V is irreducible if and only if V^* is irreducible.

Proof: If V has a proper subrepresentation U, then the inclusion $U \to V$ yields a surjection $V^* \to U^*$. Since U is non-zero and is not all of V there is a functional which is identically 0 on U but not identically 0 on V. Thus, the latter surjection has a proper kernel, which is a proper subrepresentation of V^*. On the other hand, the same argument shows that for a proper subrepresentation Λ of V^* there is $x \in V^{**}$ vanishing identically on Λ but not identically vanishing on V^*. The finite-dimensionality implies that the natural inclusion $V \subset V^{**}$ is an isomorphism. ///

[2.0.9] **Proposition:** For a G-representation V, let $J_N^* : (V_N)^* \to V^*$ be the natural dual M-map $\mu \to J_N \circ \mu$ obtained from $J_N : V \to V_N$. Then

$$J_N \circ J_N^* : (V_N)^* \to (V^*)_N$$

is an isomorphism, where the latter J_N is the map $V^* \to (V^*)_N$.

Proof: In fact, $\mu \to \mu \circ J_N$ injects $(V_N)^*$ to the subspace $(V^*)^N$ of N-fixed vectors in V^*, since for $n \in N$ and $v \in V$ we directly compute

$$(n \cdot (\mu \circ J_N))(v) \;=\; (\mu \circ J_N)(nv) \;=\; \mu(J_N(nv)) \;=\; \mu(J_N(v)) \;=\; (\mu \circ J_N)(v)$$

The N-fixed vectors $(V^*)^N$ inject to $(V^*)_N$, since for an N-fixed vector λ

$$\int_N n\lambda \, dn \;=\; \int_N \lambda \, dn \;=\; \mathrm{meas}\,(N) \cdot \lambda$$

(invoking the description above of the kernel of the quotient map to the Jacquet module). Thus, $J_N \circ J_N^*$ is an injection.

At this point we use a special feature to prove that the map is an isomorphism. Since finite-dimensional spaces are *reflexive*, apply the previous argument to V^* in place of V to obtain

$$((V^*)_N)^* \to (V^{**})_N \approx V_N$$

Generally, when $X \to Y$ is injective and $Y^* \to X^*$ is injective, both maps are isomorphisms, so we have the desired result. ///

This allows us to prove a result complementary to the earlier assertion that irreducibles V are subrepresentations of principal series if and only if $V_N \neq 0$. First, another useful property of induced representations:

[2.0.10] **Proposition:** For a finite-dimensional representation σ of a subgroup K of a finite group H, the $\mathbb{C}$-linear dual of the induced representation $\mathrm{Ind}_K^H \sigma$ is

$$\left(\mathrm{Ind}_K^H \sigma\right)^* \approx \mathrm{Ind}_K^H\,(\sigma^*)$$

via the pairing

$$\langle f, \lambda \rangle \;=\; \int_{K \backslash H} \langle f(h), \lambda(h) \rangle_\sigma \, dh$$

where $\langle , \rangle_\sigma$ is the pairing on $\sigma \times \sigma^*$.

Proof: By definition of the dual representation, the function

$$h \longrightarrow \langle f(h), \lambda(h) \rangle_\sigma$$

is left K-invariant, so gives a function on the quotient $K\backslash H$. To complete the proof we must use special features, the finiteness of H and the reflexiveness of σ. Consider functions f and λ supported on single points in $K\backslash H$, with values in dual bases of σ and σ^*. These form dual bases for the indicated induced representations. ///

[2.0.11] **Corollary:** For V an irreducible quotient of a principal series, $V_N \neq 0$ and V imbeds into a principal series.

Proof: Consider a surjection

$$\mathrm{Ind}_P^G \chi \xrightarrow{\varphi} V$$

By dualizing, and by the previous proposition, we have an injection

$$\varphi^* : V^* \to \left(\mathrm{Ind}_P^G \chi \right)^* \approx \mathrm{Ind}_P^G (\chi^*)$$

Thus, V^* imbeds into a principal series. From above, this implies that $(V^*)_N$ is non-trivial. Thus, by the isomorphism just above, $(V_N)^*$ is non-trivial. Thus, V_N must be non-trivial, so V imbeds to some principal series. ///

Thus, failure of irreducibility of I_χ gives rise to G-maps

$$I_\chi \to I_\beta$$

which are neither injections nor surjections. To study this, we have the following result, due to Mackey in the finite case, and extended by Bruhat to p-adic and Lie groups. For w in the Weyl group $W = \{1, w_o\}$ and for a character χ of M, let

$$\chi^w(m) = \chi(wmw^{-1})$$

The following result uses the finiteness of the group G.

[2.0.12] **Theorem:** The complex vectorspace $\mathrm{Hom}_G(I_\chi, I_\beta)$ of G-maps from one principal series to another is

$$\mathrm{Hom}_G(I_\chi, I_\beta) \approx \bigoplus_{w \in W} \mathrm{Hom}_M(\chi^w, \beta)$$

Generally, for two subgroups A and B of a finite group H, and for one-dimensional representations α, β of them, we have a complex-linear isomorphism

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H \alpha, \mathrm{Ind}_B^H \beta) \approx \bigoplus_{w \in A\backslash H/B} \mathrm{Hom}_{w^{-1}Aw \cap B}(\alpha^w, \beta)$$

[2.0.13] **Remark:** The decomposition over the double coset $A\backslash H/B$ is an *orbit decomposition* or *Mackey decomposition* or *Mackey-Bruhat decomposition* of the space of H-maps.

Proof: By Frobenius Reciprocity

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H \alpha, \mathrm{Ind}_B^H \beta) \approx \mathrm{Hom}_B(\mathrm{Ind}_A^H \alpha, \beta)$$

As a B-representation space, $\mathrm{Ind}_A^H\alpha$ breaks up into a sum over B-orbits on $A\backslash H$, indexed by $w \in A\backslash H/B$. Via the natural bijection

$$A\backslash AwB \to (w^{-1}Aw \cap B)\backslash B \qquad \text{by} \qquad Awb \to (w^{-1}Aw \cap B)b$$

functions on AwB with the property

$$f(awb) \;=\; \alpha(a)\, f(wb)$$

for $a \in A$ and $b \in B$ become functions on B with

$$f(b_o b) \;=\; \alpha(wb_o w^{-1})\, f(b)$$

for b_o in $w^{-1}Aw \cap B$. Thus,

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H\alpha, \mathrm{Ind}_B^H\beta) \approx \bigoplus_{w\in A\backslash H/B} \mathrm{Hom}_B(\mathrm{Ind}_{w^{-1}Aw\cap B}^B\,\alpha^w, \beta)$$

For two B-representations X and Y, there is a natural *dualization isomorphism*

$$\mathrm{Hom}_B(X, Y^*) \approx \mathrm{Hom}_B(X \otimes Y, \mathbb{C}) \approx \mathrm{Hom}_B(Y, X^*)$$

Thus, since finite-dimensional spaces are reflexive, using formulas from above for duals of induced representations,

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H\alpha, \mathrm{Ind}_B^H\beta) \approx \bigoplus_{w\in A\backslash H/B} \mathrm{Hom}_B(\beta^{-1}, \mathrm{Ind}_{w^{-1}Aw\cap B}^B\,(\alpha^w)^{-1})$$

$$\approx \bigoplus_{w\in A\backslash H/B} \mathrm{Hom}_{w^{-1}Aw\cap B}(\beta^{-1}, (\alpha^w)^{-1})$$

by Frobenius Reciprocity again. Dualizing once more,

$$\mathrm{Hom}_H(\mathrm{Ind}_A^H\alpha, \mathrm{Ind}_B^H\beta) \approx \bigoplus_{w\in A\backslash H/B} \mathrm{Hom}_{w^{-1}Aw\cap B}(\alpha^w, \beta)$$

as claimed.

For principal series representations,

$$P \cap w^{-1}Pw$$

always contains M, and we do not care what fragment of N it may or may not contain since both α and β have been extended trivially to N. This gives the assertion for principal series. ///

[2.0.14] **Corollary:** For regular χ the only G-maps of the principal series I_χ to itself are scalars.

Proof: The property that χ be regular is exactly that $\chi^w \neq \chi$. Thus, from the theorem

$$\mathrm{Hom}_G(I_\chi, I_\chi) \approx \mathrm{Hom}_M(\chi, \chi) \approx \mathbb{C}$$

since χ is one-dimensional. ///

The proof of the following corollary is contrary to the spirit of our discussion, as it invokes Complete Reducibility, but it indicates facts which we will also verify by a more generally applicable method.

[2.0.15] **Corollary:** For regular χ the principal series I_χ is irreducible and G-isomorphic to I_{χ^w}.

Proof: (Again, this is a bad proof in the sense that it uses Complete Reducibility.) For any subrepresentation W of I_χ there is a subrepresentation U such that $I_\chi = U \oplus W$. The projection to U by $u \oplus w \to u$ is a G-representation. But by the corollary above for regular χ this map must be a scalar on I_χ so either U or W is 0, proving irreducibility. Then the non-zero intertwining from I_χ to I_{χ^w} cannot avoid being an isomorphism. ///

We'll give another proof of the irreducibility of regular I_χ shortly. But at the moment we cheat in another way to count irreducibles of $G = GL(2,k)$, comparing to the number we've constructed by regular principal series.

Recall that *the number of irreducible complex representations of a finite group is the same as the number of conjugacy classes in the group.*

In $G = GL(2,k)$ with k finite with q elements, by elementary linear algebra (Jordan form) there are conjugacy classes

central	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$q-1$	of them	
non-semi-simple	$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$q-1$	of them	$(x \neq 0)$
non-central split semi-simple	$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$(q-1)(q-2)/2$	of them	$(x \neq y)$
anisotropic semi-simple	$\ldots$	$(q^2-q)/2$	of them	

where the anisotropic elements are conjugacy classes consisting of matrices with eigenvalues lying properly in the unique quadratic extension of k. Conjugation by the longest Weyl element accounts for the division by 2 in the non-central split semi-simple case. The division by 2 in the non-split semisimple accounts for the Galois action being given by a conjugation within the group.

These conjugacy classes match in an *ad hoc* fashion with specific representations. Match the central conjugacy classes with the one-dimensional representations (composing determinant with characters $k^\times \to \mathbb{C}^\times$). Match the non-semi-simple classes with the *complements* (cheating here) to the determinant representations inside the irregular principal series, called **special** representations. Match the regular principal series with non-central split semi-simple classes. Thus, numerically, there are bijections

central	$\longleftrightarrow$	one-dimensional
non-semi-simple	$\longleftrightarrow$	special
non-central split semi-simple	$\longleftrightarrow$	regular principal series
anisotropic semi-simple	$\longleftrightarrow$	supercuspidal (?!)

We assign the leftovers to supercuspidal irreducibles by default, since we have no immediate alternative for counting them. From our present viewpoint supercuspidals are defined in a negative sense as being the things for which we have no construction.

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3. *Whittaker functionals, Whittaker models*

A more extensible approach to studying the irreducibility of regular principal series representations is by distinguishing a suitable one-dimensional subspace of representations and tracking its behavior under G-maps. In fact, it turns out to be better in general to do a slightly subtler thing and distinguish a one-dimensional space of *functionals*, as follows. For a *non-trivial* character (one-dimensional representation) ψ of N, identify its representation space with $\mathbb{C}$. For a representation V of G an N-map $V \to \psi$ is a **Whittaker functional**. A **Whittaker model** for V is a (not identically 0) element of $\mathrm{Hom}_G(V, \mathrm{Ind}_N^G, \psi)$. When

$$\dim_{\mathbb{C}} \mathrm{Hom}_N(V, \psi) \;=\; 1$$

(as in the following result) one speaks of the *uniqueness of Whittaker functionals*, or *uniqueness of Whittaker models*, since Frobenius reciprocity would then give

$$\dim_{\mathbb{C}} \mathrm{Hom}_N(V, \psi) \;=\; \dim_{\mathbb{C}} \mathrm{Hom}_G(V, \mathrm{Ind}_N^G \psi)$$

[3.0.1] **Remark:** Emphasis on Whittaker functionals arose in part from consideration of Fourier expansions of modular forms.

[3.0.2] **Remark:** For $GL(2)$ the choice of non-trivial ψ does not matter since M acts transitively on non-trivial ψ:

$$\psi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}) \;=\; \psi\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

More precisely:

[3.0.3] **Proposition:** For $GL(2)$ (not $SL(2)$), with ψ and ψ' two non-trivial characters on N, there is a unique $m \in M/Z$ such that

$$\psi'(n) \;=\; \psi(mnm^{-1}) \qquad\qquad \text{(for all } n \in N\text{)}$$

Therefore, there is a G-isomorphism

$$T : \mathrm{Ind}_N^G \psi \approx \mathrm{Ind}_N^G \psi'$$

given by

$$Tf(g) \;=\; f(mg)$$

Proof: The first assertion amounts to the fact that every $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ is of the form

$$\psi(x) \;=\; \psi_o(\mathrm{tr}_{\,\mathbb{F}_q/\mathbb{F}_p}\, x)$$

where $\mathbb{F}_p$ is the prime field under $\mathbb{F}_q$ and tr is the Galois trace. [9] The formula written gives a G-map, because left multiplication by m commutes with right multiplication by g. The map is arranged to convert left equivariance by ψ into left equivariance by ψ'. ///

[3.0.4] **Remark:** For $SL(2)$ the choice of ψ *does* matter, since the number of orbits of characters on N under the M-action in that case is the cardinality of $k^\times/(k^\times)^2$, which is 2.

[3.0.5] **Proposition:** For all χ on M,

$$\dim_{\mathbb{C}} \mathrm{Hom}_N(\mathrm{Ind}_P^G\, \chi, \psi) \;=\; 1$$

Proof: By Frobenius Reciprocity and the (Mackey) orbit decomposition, as earlier,

$$\mathrm{Hom}_N(\mathrm{Ind}_P^G\, \chi, \psi) \approx \bigoplus_{w \in P\backslash G/N} \mathrm{Hom}_{w^{-1}Nw\cap N}\, \chi^w, \psi$$

For $w = 1$, since χ is trivial on N, the space of homomorphisms from χ^w to ψ is 0. Thus, there is only one non-zero summand, corresponding to the longest Weyl element $w = w_o$, and this summand gives a one-dimensional space of N-maps. ///

Given the uniqueness, an explicit formula for the Whittaker functional becomes all the more interesting, to allow normalization and comparison.

[9] This classification of characters on $\mathbb{F}_q$ is substantially a corollary of the larger fact that the *trace pairing on a finite separable extension is non-degenerate*: that is, for a finite separable field extension K/k, the symmetric k-bilinear k-valued form $\langle , \rangle$ on $K \times K$ defined by $\langle x, y\rangle = \mathrm{tr}_{\,K/k}(xy)$ is non-degenerate, in the sense that for every $x \in K$ there is $y \in K$ such that $\langle x, y\rangle \neq 0$.

[3.0.6] **Proposition:** Let w be the longest Weyl element. For $f \in \mathrm{Ind}_P^G \chi$ the formula

$$\Lambda f \;=\; \int_N f(wn)\, \overline{\psi(n)}\, dn \;\in\; \mathbb{C}$$

defines a non-zero element Λ of $\mathrm{Hom}_N(\mathrm{Ind}_P^G\, \chi, \psi)$.

Proof: It is formal, by changing variables in the integral, that the indicated expression is an N-map to ψ. To see that it is not identically 0 it suffices to see that it is non-zero on a well-chosen f. In particular, exploit the finiteness and take f to be 1 at w and 0 otherwise. Then

$$\Lambda f \;=\; \int_N f(w\nu)\, \overline{\psi(\nu)}\, d\nu \;=\; \mathrm{meas}\,\{1\} \;\neq\; 0$$

as desired. ///

[3.0.7] **Proposition:** For finite-dimensional representations V of N

$$\mathrm{Hom}_N(V, \psi) \;=\; 0 \iff \mathrm{Hom}_N(V^*, \psi^*) \;=\; 0$$

Proof: For a non-zero Whittaker functional $\Lambda \in \mathrm{Hom}_N(V, \psi)$, pick x in the second dual V^{**} such that $x(\Lambda) \neq 0$. Then

$$\Big(\int_N nx\, \psi(n)\, dn\Big)(\Lambda) \;=\; \int_N (nx)(\Lambda)\, \psi(n)\, dn \;=\; \int_N x(n^{-1}\Lambda)\, \psi(n)\, dn \;=\; \int_N x(\Lambda)\, dn \;=\; x(\Lambda) \cdot \mathrm{meas}\,(N)$$

which shows that $\int_N nx\, \psi(n)\, dn$ is not 0. For $\nu \in N$

$$\int_N n\nu \cdot x\, \psi(n)\, dn \;=\; \psi^*(\nu) \int_N n \cdot x\, \psi(n)\, dn$$

by replacing n by $n\nu^{-1}$. ///

We can use the Whittaker functionals Λ to redo our study of intertwinings $T : I_\chi \to I_{\chi^w}$ among principal series.

[3.0.8] **Proposition:** A finite-dimensional representation V of G with

$$\mathrm{Hom}_N(V, \psi) \;=\; \mathbb{C}$$

and with

$$\mathrm{Hom}_N(V^*, \psi^*) \;=\; \mathbb{C}$$

is irreducible if and only if the Whittaker functional in $\mathrm{Hom}_N(V, \psi)$ generates the dual V^* and the Whittaker functional in $\mathrm{Hom}_N(I_\chi^*, \psi^*)$ generates the second dual $V^{**} \approx V$.

Proof: On one hand, if V is irreducible, then (from above) the dual is irreducible, so certainly is generated (under G) by the (non-zero) Whittaker functional. The same applies to the second dual. This is the easy part of the argument. On the other hand, suppose that the Whittaker functionals generate (under G) the dual and second dual. A proper subrepresentation Λ of V^* cannot contain the Whittaker functional, since the Whittaker functional generates the whole representation. Thus, the image of the Whittaker functional in the quotient $Q = V^*/\Lambda$ is not 0. From just above, since Q contains a non-zero Whittaker functional so must Q^* (for the character ψ^*). But then the natural inclusion

$$Q^* \subset V^{**}$$

shows that the Whittaker vector generates a proper subrepresentation of V^{**}, contradiction. ///

[3.0.9] **Proposition:** The dual I_χ^* of a principal series I_χ fails to be generated by a Whittaker functional Λ if and only if there is a non-zero intertwining $T : I_\chi^* \to I_{\chi^w}^*$ in which $T\Lambda = 0$, for some w in the Weyl group W.

Proof: If Λ generates a proper subrepresentation V of I_χ^*, then there is an irreducible non-zero quotient Q of I_χ^*/V. From above, Q again imbeds into some principal series I_ω. This yields a non-zero intertwining $I_\chi^* \to I_\omega$ in which the Whittaker functional is mapped to 0. ///

Recall that the only principal series representation admitting a non-zero intertwining from I_χ is I_{χ^w}, for w in the Weyl group. For a character $\omega : k^\times \to \mathbb{C}^\times$, define a Gauss sum

$$g(\omega, \psi) \;=\; \int_{k^\times} \omega(x)\, \overline{\psi} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} dx$$

where the measure gives each element of $k^\times$ measure 1. Let

$$\Lambda_\chi f \;=\; \int_N f(w_o n)\, \psi(n)\, dn$$

be the normalized Whittaker functional in I_χ^*.

[3.0.10] **Proposition:** Let $w = w_o$ be the longest Weyl element. Under the intertwining $T : I_\chi \to I_{\chi^w}$ given by

$$Tv(g) \;=\; \int_N v(wng)\, dn$$

the normalized Whittaker functional Λ_{χ^w} in $(I_{\chi^w})^*$ is mapped by the adjoint T^* to

$$T^*(\Lambda_{\chi^w}) \;=\; g(\chi, \psi) \cdot \Lambda_\chi \;\in\; (I_\chi)^*$$

Proof: Using the uniqueness of the Whittaker functionals on principal series, it suffices to compute the values of the images on a well-chosen function f.

$$\Lambda^w T f \;=\; \int_N Tf(wn)\, \overline{\psi(n)}\, dn \;=\; \int_N \int_N f(w\nu w n)\, \overline{\psi(n)}\, d\nu\, dn$$

To compare this to

$$\Lambda f \;=\; \int_N f(wn)\, \overline{\psi(n)}\, dn$$

take f to be

$$f(nmw) \;=\; \chi(m)$$

for $n \in N$ and $m \in M$, and 0 otherwise. That is, f is supported on Pw and is 1 at w. Then $wn \in Pw$ if and only if $n = 1$. Thus,

$$\Lambda f \;=\; \overline{\psi(1)} \cdot \text{meas}\,\{1\} \;=\; \text{meas}\,\{1\}$$

On the other hand, the condition $w\nu wn \in Pw$ is met in a more complicated manner. Indeed, letting $\nu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we have the standard identity for $x \neq 0$

$$w\nu w \;=\; \begin{pmatrix} -1 & 0 \\ x & -1 \end{pmatrix} \;=\; \begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix}$$

Note that this identity works in both $GL(2)$ and in $SL(2)$. Thus, $w\nu wn \in Pw$ if and only if

$$n = \begin{pmatrix} 1 & 1/x \\ 0 & 1 \end{pmatrix}$$

and in that case

$$w\nu wn = \begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Thus, for $G = GL(2)$

$$\Lambda_{\chi^w} Tf = \text{meas}\{\cdot\} \int_{k^\times} \frac{\chi_2}{\chi_1}(x)\, \psi^* \begin{pmatrix} 1 & 1/x \\ 0 & 1 \end{pmatrix} dx$$

where meas $\{\cdot\}$ is the measure of a singleton. Replacing x by $1/x$, since meas $\{\cdot\}$ = meas $\{1\}$, we conclude that with the intertwining

$$Tv(g) = \int_N v(wng)\, dn$$

the normalized Whittaker functional Λ_{χ^w} is mapped to $g(\chi, \psi)$ times the Whittaker functional Λ_χ under the adjoint T^*. For $G = SL(2)$ the conclusion is nearly identical, with χ_2 replaced by χ_1^{-1}, in effect. ///

[3.0.11] Corollary: For χ regular the corresponding Gauss sum is non-zero, hence the Whittaker functional is never annihilated by a non-zero intertwining, hence the Whittaker functional generates I_χ^*. Likewise the corresponding Whittaker functional generates I_χ^{**}. Thus, I_χ is irreducible.

Proof: We recall a computation that proves the Gauss sum is non-zero for $\chi_1 \neq \chi_2$. Let the measure on $k^\times$ give singleton sets measure 1.

$$|g(\chi, \psi)|^2 = \int_{k^\times} \int_{k^\times} \frac{\chi_1}{\chi_2}(x/y)\, \overline{\psi} \begin{pmatrix} 1 & x-y \\ 0 & 1 \end{pmatrix} dx\, dy = \int_{k^\times} \int_{k^\times} \frac{\chi_1}{\chi_2}(x)\, \overline{\psi} \begin{pmatrix} 1 & y(x-1) \\ 0 & 1 \end{pmatrix} dx\, dy$$

replacing x by xy. For fixed $x \neq 1$, the integral over y would be over $k^\times$ if it were not missing the $y = 0$ term, so it is

$$\int_k \overline{\psi} \begin{pmatrix} 1 & y(x-1) \\ 0 & 1 \end{pmatrix} dy - 1 = -1$$

For $x = 1$, the integral is $q - 1$, where $|k| = q$. Thus,

$$|g(\chi, \psi)|^2 = q - \int_{k^\times} \frac{\chi_1}{\chi_2}(x)\, dx = q - 0$$

for $\chi_1 \neq \chi_2$. Thus, the Gauss sum is non-zero. Thus, the adjoint T^* of the intertwining $T : I_\chi \to I_{\chi^w}$ just above does not annihilate the Whittaker functional. Since χ is regular, χ^w is regular, and every non-zero intertwining of I_χ to itself is a non-zero multiple of the identity, so again the Whittaker functional is not annihilated by the adjoint T^*.

Thus, the Whittaker functional generates the dual I_χ^*. Similarly, the corresponding Whittaker functional generates the second dual, and from above we conclude that I_χ is irreducible. ///

[3.0.12] Remark: The previous discussion is a simple example illustrating the spirit of Casselman's 1980 use of *spherical* vectors to examine irreducibility of unramified principal series of p-adic reductive groups.

[3.0.13] Remark: For irregular χ we could invoke complete reducibility and the computation (above) that for irregular χ

$$\dim_{\mathbb{C}} \text{Hom}_G(I_\chi, I_\chi) = \text{card}\, P\backslash G/P = 2$$

to see that I_χ is a direct sum of two irreducibles. Further, we can immediately identify the one-dimensional subrepresentation $\chi_1 \circ \det$ of I_χ for irregular $\chi = (\chi_1, \chi_1)$ for $GL(2)$. It is immediate that $\chi_1 \circ \det$ has no

Whittaker functional, so we can anticipate that (still using complete reducibility) the *other* irreducible in irregular I_χ has a Whittaker functional. This other irreducible is a *special* representation.

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4. *Uniqueness of Whittaker functionals/models*

So far we have no tangible description for the supercuspidal irreducibles V, by definition those for which $V_N = 0$. In particular, we cannot address uniqueness of Whittaker functionals for supercuspidals by explicit computation since we have no tangible models for these irreducibles, but their Whittaker models exist simply because the Jacquet modules are trivial (see just below). We can note that, for a representation V of G, by Frobenius Reciprocity

$$\mathrm{Hom}_N(V, \psi) \approx \mathrm{Hom}_G(V, \mathrm{Ind}_N^G \psi)$$

That is, Whittaker functionals correspond to G-intertwinings to the *Whittaker space* $\mathrm{Ind}_N^G \psi$.

[4.0.1] **Proposition:** A supercuspidal irreducible V of $GL(2)$ has a Whittaker model.

Proof: As a representation of N (by restriction), V is a sum of irreducibles. Since V is supercuspidal its Jacquet module is trivial and the trivial representation of N does not occur. Thus, a non-trivial representation ψ of N does occur. Since N is abelian, this irreducible is one-dimensional. Then, since V is stable under that action of M, and (as observed earlier) M is transitive on non-trivial characters on N, *every* non-trivial ψ of N occurs in V. ///

[4.0.2] **Remark:** The analogous result about Whittaker models for supercuspidal representations is more complicated for $SL(2)$, as we will see later.

[4.0.3] **Theorem:** Let ψ be a non-trivial character on N. The endomorphism algebra

$$\mathrm{Hom}_G(\mathrm{Ind}_N^G \psi, \mathrm{Ind}_N^G \psi)$$

is *commutative*. Thus, we have *Uniqueness of Whittaker functionals* : For an irreducible representation V of G

$$\dim_{\mathbb{C}} \mathrm{Hom}_N(V, \psi) \le 1$$

Equivalently, we have *Uniqueness of Whittaker models*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V, \mathrm{Ind}_N^G \psi) \le 1$$

[4.0.4] **Remark:** For $GL(2)$, the only case where the dimension of intertwinings is 0 rather than 1 is for the one-dimensional representations, that is, for composition of determinant with characters of $k^\times$.

Proof: First, we see how commutativity of the endomorphism ring implies that multiplicities are ≤ 1. Use complete reducibility, so

$$\mathrm{Ind}_N^G \psi \approx \bigoplus_V m_V \cdot V$$

where V runs through isomorphism classes of irreducibles and m_V is the multiplicity of V. Then

$$\mathrm{End}_G(\mathrm{Ind}_N^G \psi) \approx \prod_V M_{m_V}(\mathbb{C})$$

where $M_n(\mathbb{C})$ is the ring of n-by-n matrices with complex entries. Thus, this endomorphism ring is commutative if and only if all the multiplicities are 1.

To study the endomorphism ring, use the Mackey-Bruhat orbit decomposition of the space of intertwinings from one induced representation to another in the case that the two induced representations are the same. Thus, given

$$T \in \mathrm{Hom}_G(\mathrm{Ind}_A^G \alpha, \mathrm{Ind}_B^G \beta)$$

let K_T be a function on $G \times G$ such that

$$Tf(g) \;=\; \int_G K_T(g,h)\, f(h)\, dh$$

The fact that T is a G-map gives, for all $x \in G$,

$$\int_G K_T(gx,h)\, f(h)\, dh \;=\; Tf(gx) \;=\; (R_x Tf)(g)$$

$$=\; (TR_x f)(g) \;=\; \int_G K_T(g,h)\, f(hx)\, dh \;=\; \int_G K_T(g,hx^{-1})\, f(h)\, dh$$

by replacing h by hx^{-1}, where R_x is the right regular representation. Thus, the kernel K_T is just a function of a single variable, and the intertwining T can be rewritten as

$$Tf(g) \;=\; \int_G K_T(gh^{-1})\, f(h)\, dh$$

Since T maps to $\mathrm{Ind}_B^G \beta$, and maps from $\mathrm{Ind}_A^G \alpha$, it must be that

$$K_T(bxa) \;=\; \beta(b) \cdot K_T(x) \cdot \alpha(a)$$

for all $b \in B$, $g \in G$, $a \in A$. A direct computation shows that

$$K_{S\circ T} \;=\; K_S * K_T$$

for $S, T \in \mathrm{Hom}_G(\mathrm{Ind}_A^G \alpha, \mathrm{Ind}_B^G \beta)$ where the convolution is as usual

$$(f * \varphi)(g) \;=\; \int_G f(gx^{-1})\, \varphi(x)\, dx$$

Thus, to prove commutativity of the endomorphism ring it is necessary and sufficient to prove commutativity of the convolution ring R of complex-valued functions u on G with the equivariance properties

$$u(bxa) \;=\; \beta(b) \cdot u(x) \cdot \alpha(a)$$

for $a \in A$, $b \in B$, $x \in G$.

Note that, for $A = B$ and $\alpha = \beta$, the convolution of two such functions falls back into the same class.

Following Gelfand-Graev and others, to prove commutativity of such a convolution ring, it suffices to find an involutive anti-automorphism σ of G such that for u on G with the property

$$u(bxa) \;=\; \psi(b) \cdot u(x) \cdot \psi(a)$$

we have

$$u(g^\sigma) \;=\; u(g) \qquad \text{(for all } g \in G\text{)}$$

To verify that this criterion for commutativity really works, use notation

$$u^\sigma(g) \;=\; u(g^\sigma)$$

and let u, v be two such functions. Then

$$(u^\sigma * v^\sigma)(x) \;=\; \int_G u((xg^{-1})^\sigma)\, v(g^\sigma)\, dg \;=\; \int_G u(g\, x^\sigma)\, v(g^{-1})\, dg$$

by replacing g by $(g^\sigma)^{-1}$. Replacing g by $g(x^\sigma)^{-1}$ turns this into

$$\int_G u(g)\, v(x^\sigma g^{-1})\, dg \;=\; (v * u)^\sigma(x)$$

That is,

$$(u * v)^\sigma \;=\; v^\sigma * u^\sigma$$

Therefore, if $u = u^\sigma$ and $v = v^\sigma$ then

$$u * v \;=\; (u * v)^\sigma \;=\; v^\sigma * u^\sigma \;=\; v * u$$

and the convolution ring is commutative.

To apply the Gelfand-Graev involution idea, we need to classify functions u such that, as above,

$$u(bxa) \;=\; \psi(b) \cdot u(x) \cdot \psi(a)$$

since these are the ones that could occur as Mackey-Bruhat kernels. We use the *Bruhat decomposition*, namely that

$$G \;=\; NM \cup NMwN \qquad \text{(disjoint union, with } w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{)}$$

where M is diagonal matrices and w is a slightly different normalization of longest Weyl element. The group M normalizes N, but does not preserve ψ, since for $m \in M$

$$\psi(mnm^{-1}) \;\neq\; \psi(n)$$

for all $n \in N$ unless m is actually in the center Z of G, the scalar matrices. The two-sided equivariance condition entails

$$\psi(n)u(m) \;=\; u(nm) \;=\; u(m \cdot m^{-1}nm) \;=\; u(m)\, \psi(m^{-1}nm)$$

This does not hold for all $n \in N$ unless m is central. Thus, the left and right N, ψ-equivariant functions supported on NM are those whose support is NZ. For the equivariant functions supported on $NMwN$, there is no such issue, since $N \cap wNw^{-1} = \{1\}$. Thus, the $N \times N$ orbits which can support such equivariant functions are those with representatives $z \in Z$ and mw with $m \in M$. All such functions are linear combinations of functions

$$f(nz) \;=\; \psi(n) \qquad \text{(for } n \in N\text{, 0 otherwise)}$$

for fixed $z \in Z$ and, for fixed $m \in M$,

$$f(nmw\nu) \;=\; \psi(n)\psi(\nu) \qquad \text{(for } n, \nu \in N\text{, 0 otherwise)}$$

In this situation, with w the long Weyl element normalized as above, take involutive anti-automorphism

$$g^\sigma \;=\; wg^\top w^{-1}$$

This is the identity on the center Z, on N, and on elements mw with $m \in M$. Thus, it is the identity on all such equivariant functions. Thus, this convolution ring meets the Gelfand-Graev criterion for commutativity. ///

[4.0.5] **Remark:** The kernels K_T introduced in the proof have analogues in more complicated settings, and would more generally be called *Mackey-Bruhat distributions*. That is, the relevant kernel would not in general be given by a *function*, but by a Schwartz *distribution*.

[4.0.6] **Remark:** Without complete reducibility, the principle that commutativity of an endomorphism ring implies that multiplicities are all ≤ 1 acquires a more complicated form. One version is the *Gelfand-Kazhdan criterion*. The same approach, namely Mackey-Bruhat and Gelfand-Graev, yield further facts whose analogues are more complicated over non-finite fields.

[4.0.7] **Remark:** For *irregular* χ, we have already seen that I_χ decomposes as the direct sum of two non-isomorphic irreducibles. Thus, for given ψ, one of these subrepresentations has a Whittaker model and one does not. For $GL(2)$, the irregular principal series always have one-dimensional subrepresentation, which fails to have a Whittaker model. For $SL(2)$, it is less clear.

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5. *Summary for* $GL(2)$

There are $(q-1)(q-2)/2$ isomorphism classes of irreducible principal series (with $I_\chi \approx I_{\chi^w}$), namely the *regular* ones (i.e., with $\chi^w \neq \chi$). These all have Whittaker models. Their Jacquet modules are 2-dimensional. They are of dimension $|P\backslash G| = q+1$.

There are $q-1$ one-dimensional representations, obtained by composing characters with determinant. Their Jacquet modules are 1-dimensional, not surprisingly. These do not have Whittaker models.

There are $q-1$ *special* representations, subrepresentations of irregular principal series. Their Jacquet modules are 1-dimensional. They have Whittaker models (since every unramified principal series has a Whittaker functional and one-dimensional representations do not admit such.) Special representations are of dimension q.

There are $q(q-1)/2$ supercuspidal irreducibles, by definition having 0-dimensional Jacquet module, all having a Whittaker model. Each has dimension $q-1$.

[5.0.1] **Remark:** Since M is transitive on non-trivial characters on N, there is (up to G-isomorphism) only *one* Whittaker space. This is not true for $SL(2)$.

[5.0.2] **Remark:** One numerical check for the above categorization is the fact (from decomposition of the biregular representation) that the sum of the squares of the dimensions of the irreducibles is the order of the group. Thus, we should have (in the same order that we reviewed them)

$$(q^2-1)(q^2-q) \;=\; (\text{cardinality of } GL(2) \text{ over field with } q \text{ elements})$$

$$=\; (\text{irreducible principal series}) + (\text{one-dimensional}) + (\text{special}) + (\text{supercuspidal})$$

$$=\; \frac{(q-1)(q-2)}{2}\cdot(q+1)^2 + (q-1)\cdot 1^2 + (q-1)\cdot q^2 + \frac{q(q-1)}{2}\cdot(q-1)^2$$

Remove a factor of $q-1$ from both sides, leaving a supposed equality

$$(q^2-1)q \;=\; \frac{(q-2)}{2}\cdot(q+1)^2 + 1 + q^2 + \frac{q(q-1)}{2}\cdot(q-1)$$

Anticipating a factor of q throughout, combine the first two summands on the right-hand side to obtain (multiplying everything through by 2, as well)

$$2(q^2-1)q \;=\; \left(q^3-3q\right) + 2q^2 + q(q-1)^2$$

which allows removal of the common factor of q, to have the supposed equality

$$2(q^2-1) \;=\; q^2 - 3 + 2q + (q-1)^2$$

The degree is low enough to multiply out, giving an alleged equality

$$2q^2 - 2 \ = \ q^2 - 3 + 2q + q^2 - 2q + 1$$

which is easy to verify. The reduction steps were reversible, so this counting check succeeds.

[5.0.3] **Remark:** Another numerical check would be by counting the irreducibles with Whittaker models, versus the dimension of the space of endomorphisms of the Whittaker space, since the latter is commutative (above). The number of irreducibles with Whittaker models is

$$\text{(irreducible principal series)} + \text{(special)} + \text{(supercuspidal)}$$

$$= \ \frac{(q-1)(q-2)}{2} + (q-1) + \frac{q(q-1)}{2} \ = \ (q-1)\left[\frac{q-2}{2} + 1 + \frac{q}{2}\right] \ = \ q(q-1)$$

On the other hand, the dimension of the space of endomorphisms of the Whittaker space (from the proof of commutativity of the endomorphism ring, above) is the cardinality

$$\text{(number of left-and-right } N \times N \text{ orbits supporting left-and-right } \psi\text{-equivariant functions)}$$

$$= \ \mathrm{card}\,(N\backslash NZ/N \sqcup N\backslash Pw_oP/N) \ = \ \mathrm{card}\,(Z) + \mathrm{card}\,(M) \ = \ (q-1) + (q-1)^2 \ = \ q(q-1)$$

where Z is the center of $GL(2)$. They match.

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6. *Conjugacy classes in $SL(2)$, odd q*

Before pairing up conjugacy classes and irreducibles for $SL(2)$ over a finite field with q elements, we must take greater pains to identify conjugacy classes. For $SL(2)$ the parity of q matters, while it did not arise for $GL(2)$. In $G = SL(2,k)$ with k finite with q elements, the collection of conjugacy classes is more complicated than the pure linear algebra of $GL(2,k)$. The non-semisimple elements' conjugacy classes are most disturbed by the change from $GL(2)$ to $SL(2)$. Let ϖ be a non-square in $k^\times$, and take q *odd*.

central	$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	2	of them	$(x = \pm 1)$
non-semisimple	$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	2	of them	$(x = \pm 1)$
non-semisimple	$\begin{pmatrix} x & \varpi \\ 0 & x \end{pmatrix}$	2	of them	$(x = \pm 1)$
non-central split semi-simple	$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$	$(q-3)/2$	of them	$(x \neq \pm 1)$
non-split semi-simple	$\ldots$	$(q-1)/2$	of them	

where the anisotropic elements are conjugacy classes of matrices with eigenvalues lying properly in the (unique) quadratic extension of k, and with Galois norm 1. The division by 2 in the latter is because the Galois action is given by a conjugation in the group. In the case of split semi-simple elements the division by 2 reflects the fact that conjugation interchanges a and a^{-1}. Verification that these are exactly the $SL(2)$ conjugacy classes is at least mildly interesting, and we carry out this exercise to have specifics used later.

Sketch the discussion for odd q. First, observe that if $g \in G$ has elements in its centralizer $C(g)$ in $GL(2)$ having determinants running through all of $k^\times$, then

$$\{xgx^{-1} : x \in SL(2)\} \ = \ SL(2) \cap \{xgx^{-1} : x \in GL(2)\}$$

That is, with the hypothesis on the centralizer, the intersection with $SL(2)$ of a $GL(2)$ conjugacy class does not break into proper subsets under $SL(2)$ conjugation. For g central or of the form $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ the element $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$ in the centralizer has determinant $d \in k^\times$, meeting this hypothesis. Now consider non-split semi-simple elements g. It is elementary that such g lies in an imbedded copy of the norm-one elements K^1 in the unique quadratic extension K of k. The group $K^\times$ imbeds compatibly in $GL(2)$, and determinant on the imbedded copy is the Galois norm. Since norm is surjective on finite fields, non-split semi-simple conjugacy classes also meet the hypothesis above, so there is no change from $GL(2)$ to $SL(2)$.

The non-semi-simple classes are subtler. First, non-semisimple elements u must have rational eigenvalues, and the non-semi-simplicity then implies that such u stabilizes a unique line λ in k^2. By the transitivity of $SL(2)$ on lines, all non-semi-simple conjugacy classes in $SL(2)$ have representatives of the form $u = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ with non-zero upper with $a = \pm 1$, stabilizing the obvious line λ. If another such matrix $v = \begin{pmatrix} b & * \\ 0 & b^{-1} \end{pmatrix}$ with non-zero upper right entry is conjugate to u, say $x^{-1}vx = u$, then $vx = xu$ and

$$vx \cdot \lambda \;=\; xu \cdot \lambda$$

from which

$$vx \cdot \lambda \;=\; x \cdot \lambda$$

since u fixes λ. This implies that v fixes $x\lambda$, so $x\lambda = \lambda$ (since v fixes exactly one line), and necessarily x is of the form

$$x \;=\; \begin{pmatrix} b & * \\ 0 & b^{-1} \end{pmatrix}$$

for some $b \in k^\times$. By this point, the remaining computations are not hard. Specifically, conjugation by upper-triangular matrices in $SL(2)$ acting on matrices $\begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix}$ adjusts the upper-right entry only by squares in $k^\times$. Since $k^\times$ is cyclic, there are exactly two orbits. Thus, as asserted above, the non-semi-simple conjugacy classes have representatives

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & \varpi \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & \varpi \\ 0 & -1 \end{pmatrix}$$

where ϖ is a non-square in $k^\times$.

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7. *Irreducibles of* $SL(2)$, q *odd*

Now we classify irreducibles of $G = SL(2)$ over a finite field with an *odd* number of elements q. Unlike the case of $GL(2)$, for $SL(2)$ there are *two* inequivalent families of Whittaker models, as there are two characters ψ and ψ' on N, *not* related to each other by conjugation by M, unlike $GL(2)$. Fix two such $SL(2)$-unrelated ψ and ψ', and refer to the ψ-Whittaker and ψ'-Whittaker models or functionals.

First, parallel to the discussion of principal series for $GL(2)$, the principal series

$$I_\chi \;=\; \mathrm{Ind}_P^G\,\chi$$

for the $q-3$ *regular* χ's on M are *irreducible*, and there is an isomorphism

$$I_\chi \to I_{\chi^{-1}}$$

so there are $(q-3)/2$ irreducibles occurring as principal series. There are exactly two *irregular* characters here, the trivial character and the (unique) other character that assumes values ± 1. Let the corresponding principal series be denoted I_1 and I_{-1}. Just as for $GL(2)$

$$I_1 \;=\; \mathbb{C} \oplus \text{special}$$

where $\mathbb{C}$ is the trivial representation. The same techniques show that

$$I_{-1} \;=\; \text{direct sum of two irreducibles}$$

but neither of the two irreducibles is one-dimensional. Both of these have one-dimensional Jacquet modules, since they both imbed into a principal series.

[7.0.1] **Remark:** At least for $q \geq 3$ one can show that the derived group of $G = SL(2, \mathbb{F}_q)$ is G itself, so there can be no non-trivial one-dimensional representations of G.

It remains true for $SL(2)$ that for either Whittaker model, ψ or ψ', there is a unique Whittaker functional on a (regular or not) principal series I_χ. The trivial representation has no Whittaker model of *either* type, so the *special* representation has a Whittaker model of *both* types. Irreducible principal series have Whittaker models of *both* types.

The nature of the Whittaker models (or lack thereof) is not clear yet for the irreducibles into which the irregular I_{-1} decomposes.

[7.0.2] **Proposition:** A supercuspidal irreducible for $SL(2)$ has *either* a ψ-Whittaker model or a ψ'-Whittaker model.

Proof: A supercuspidal, which by definition has a trivial Jacquet module, must have a non-trivial ψ-isotype for N for *some* ψ. As observed in the discussion of the Whittaker spaces for $GL(2)$, conjugation by M gives G-isomorphic Whittaker spaces. Thus, if ψ and ψ' are representatives for the two M-orbits, a supercuspidal must have one or the other Whittaker model. ///

[7.0.3] **Proposition:** The number of irreducibles of $SL(2)$ with ψ-Whittaker models is $q+1$. The number of irreducibles with ψ'-Whittaker models is $q+1$. The number irreducibles which have *both* types of Whittaker models is $q-1$.

Proof: The argument used in the $GL(2)$-case, following Mackey-Bruhat and Gelfand-Graev, succeeds here. The support of a left and right ψ-equivariant distribution on $SL(2)$ must have support on

$$NZ \sqcup NMw_oN$$

and (keeping in mind that q is odd) the dimension of the space of all such is the cardinality

$$\text{card } N\backslash(NZ \sqcup NMw_oN)/N \;=\; 2 + (q-1) \;=\; q+1$$

The same conclusion works for any non-trivial character. If, instead, we require left ψ'-equivariance and right ψ-equivariance with M-inequivalent characters, we claim that only the larger Bruhat cell can support appropriate distributions, so the dimension is $q-1$. Indeed, this is exactly the assumption that ψ and ψ' are not conjugated to each other by any element of the Levi component M in $SL(2)$. ///

Thus, the two non-isomorphic types of Whittaker models have exactly $q-1$ isomorphism classes in common out of $q+1$ in each. The $(q-3)/2$ irreducible principal series account for some of these common ones. The *special* representation (in I_1) is another that lies in both, since the trivial representation lies in *neither*, and I_1 has a unique Whittaker vector (for either character).

[7.0.4] **Lemma:** One of the two irreducible summands of I_{-1} lies in one Whittaker space and the other lies in the other Whittaker space.

Proof: When an irreducible V has non-trivial

$$\psi : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \to \psi_o(x)$$

isotype for N, under the action of M it also has a non-trivial

$$\psi_a : \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \to \psi_o(a^2 x)$$

isotype for N. There are $(q-1)/2$ characters in such an M-orbit. Thus, if ψ and ψ' are M-inequivalent and V has *both* ψ-Whittaker and ψ'-Whittaker models, V has a non-trivial isotype for *all* of the $q-1$ non-trivial characters on N. As remarked above, the two summands in I_{-1} both have one-dimensional Jacquet modules (trivial N-isotypes), and are not one-dimensional. Thus, the dimension of each summand in I_{-1} is at least

$$1 + (q-1)/2 \ = \ (q+1)/2$$

The dimension of the whole I_{-1} is $q+1$, so it must be that each has dimension *exactly* $(q+1)/$. Thus, indeed, one has one type of Whittaker model, and the other has the other type. ///

So far, each Whittaker space has the unique special representation (from I_1), $(q-3)/2$ irreducible principal series, and 1 from among the two summands of I_{-1}. Each supercuspidal irreducible has at least one Whittaker model from among the two. Only the trivial (one-dimensional) representation has *no* Whittaker model of either type.

The previous proposition shows that there are 4 irreducibles with *exactly one* Whittaker model, and that two of these have a ψ-model and two have a ψ'-model. The two irreducible summands of I_{-1} account for two of these. The remaining irreducibles are (by definition) supercuspidal. Thus, there are exactly 2 supercuspidal irreducibles of $SL(2)$ having a *single type* of Whittaker model.

We can do a numerical check. Again, the number of conjugacy classes in $SL(2)$ over a field with an odd number q of elements is

(central)+(non-semi-simple)+(new non-semi-simple)+(non-central split semisimple)+(non-split semisimple)

$$= \ 2 + 2 + 2 + \frac{(q-3)}{2} + \frac{(q-1)}{2} \ = \ q+4$$

Thus, excluding the trivial representation, there are $q+3$ irreducibles *with* at least one type of Whittaker model. There are $q-1$ irreducibles in common between the two types of Whittaker models, and each model has dimension $q+1$, so the total indeed is

$$2 \cdot (q+1) - (q-1) \ = \ q+3$$

[7.0.5] **Remark:** Remarks just above also show that the supercuspidal irreducibles with *both* types of Whittaker models are of dimension $q-1$ (the number of all non-trivial characters of N), while the 2 supercuspidal irreducibles with only *one* type of model are of dimension $(q-1)/2$, distinguishing these two smaller supercuspidals among supercuspidals.