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A stunt using traces

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Specific choices of physical objects on which to do harmonic analysis can be enlightening. With $\Delta u = -u''$, consider a very simple Sturm-Liouville problem:

$$\Delta u \;=\; f \text{ on } [a,b] \text{ with } u(a) = u(b) = 0$$

A *Green's function*[1] for this problem is[2]

$$G(x,y) \;=\; \begin{cases} (y-a)(b-x)/(b-a) & (\text{for } a \le y < x \le b) \\ (x-a)(b-y)/(b-a) & (\text{for } a \le x < y \le b) \end{cases}$$

The associated eigenvalue problem,

$$(\Delta - \lambda)u \;=\; 0$$

with the same boundary conditions, specialized to $a = 0$ and $b = 1$, is easily solved directly, yielding eigenvectors

$$u_n(x) \;=\; \sin(\pi n x) \qquad\qquad (\text{for } n \ge 1)$$

The *trace* of the *inverse* mapping

$$T : f \;\longrightarrow\; \int_0^1 G(x,y)\, f(y)\, dy$$

can be evaluated two ways: sum the inverses of the eigenvalues for the differential operator, and as the integral along the diagonal,[3] $\int_a^b G(x,x)\,dx$ of the kernel

$$G(x,y) \;=\; \begin{cases} y(x-1) & (\text{for } 0 \le y < x \le 1) \\ x(y-1) & (\text{for } 0 \le x < y \le 1) \end{cases}$$

Thus,

$$\sum_{n\ge1} \frac{1}{(\pi n)^2} \;=\; \int_0^1 x(1-x)\,dx \;=\; \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

One can also evaluate $\zeta(2k)$ by computing the iterated kernel and taking its trace. For $\zeta(4)$ this is still not too unpleasant.

Naturally, one should have some care about taking traces of operators.

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[1] We are *not* appealing to any apocryphal existence argument for Green's functions in general.

[2] To be annihilated by Δ (in x) away from $x = y$, for fixed y $f(x) = G(x,y)$ is piecewise linear, say $f(x) = A(x-a)$ for $a \le x < y$ and $f(x) = B(x-b)$ for $y < x \le b$, where A and B depend upon y. So that $f(x)$ is continuous at $x = y$ these two linear fragments must match at $x = y$, so $A(y-a) = B(y-b)$. The first derivative in x is then A to the left of y and B to the right. For the negative second derivative to be δ, $A - B = 1$. Solving for A and B gives the indicated $G(x,y)$. Application of Δ to $G(x,y)$ in x gives a δ at y as desired, but also multiples of δ at the endpoints a, b. Thus, the problem is posed on the space of functions vanishing at the endpoints. The minor conundrum is that vanishing at endpoints does not make sense in $L^2(0,1)$.

[3] That the trace is the integral of the integral kernel along the diagonal is not trivially proven. Expressing the operator T as a limit of finite-rank operators allowing an analogous computation of trace is one argument that this trace exists and that the diagonal integral computes it.