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Integrals of products of eigenfunctions on $SL_2(\mathbb{C})$

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We determine precise asymptotics in spectral parameters for integrals of n-fold products of zonal spherical harmonics on $SL_2(\mathbb{C})$.

In a variety of situations, integrals of products of eigenfunctions have faster decay than smoothness entails. This phenomenon does not appear for abelian or compact groups, since irreducibles are finite-dimensional, so the decomposition of a tensor product of irreducibles is *finite*. In contrast, for non-compact, non-abelian groups irreducibles are typically infinite-dimensional, and the decomposition of a tensor product of irreducibles typically contains infinitely-many irreducibles, and the issue is non-trivial.

In a few situations integrals of products of eigenfunctions are elementary, and these deserve attention. Zonal spherical functions for complex reductive groups are elementary, and the rank-one case $SL_2(\mathbb{C})$ is the simplest.

The automorphic version of asymptotics for integrals of products of eigenfunctions is more delicate, and more complicated, as illustrated by [Sarnak 1994]. [Bernstein-Reznikoff 1999] and [Krotz-Stanton 2004] show that the critical mechanism for exponential decay is extendability of matrix coefficient functions to *complexifications* of Lie groups or symmetric spaces. This phenomenon is well-known for Fourier transforms on Euclidean spaces.

The example of $SL_2(\mathbb{C})$ is less trivial than the Euclidean case, but still transparent. The references indicate sources for spherical functions more generally.

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1. *Zonal spherical harmonics on* $SL_2(\mathbb{C})$

We review some standard facts and set up notation. Let $G = SL_2(\mathbb{C})$ and $K = SU(2)$. The standard *split component* is

$$A^+ \;=\; \{a_r = \begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix} \;:\; r \geq 0\}$$

The Cartan decomposition is

$$G \;=\; KA^+K$$

In Cartan coordinates, the Haar measure on G is

$$\text{Haar measure} \;=\; d(ka_rk') \;=\; |\sinh r|^2\, dk\, dr\, dk' \;=\; \sinh^2 r\, dk\, dr\, dk'$$

The **Laplacian** Δ on G/K is the restriction of the Casimir operator Ω, and on left-and-right K-invariant functions is a constant multiple of

$$f \;\longrightarrow\; f''(r) + 2\coth r \cdot f'(r) \qquad\qquad (\text{on functions } F(ka_rk') = f(r))$$

A **zonal spherical function** on G is a smooth K-bi-invariant eigenfunction for Δ, that is, a function F such that

$$\Delta F \;=\; \lambda \cdot F$$

The standard normalization is that a zonal spherical function takes value 1 at $1 \in G$, that is, at $r = 0$. Thus, put

$$\varphi_s(r) \;=\; \frac{\sinh(2s-1)r}{(2s-1)\,\sinh r} \qquad\qquad \text{(zonal spherical function, eigenvalue } s(s-1))$$

On the unitary line $s = \frac{1}{2} + it$, the spherical function is

$$\varphi_{\frac{1}{2}+it}(r) \;=\; \frac{\sin 2tr}{2t\,\sinh r} \;=\; \varphi_{\frac{1}{2}-it}(r) \qquad\qquad \text{(zonal spherical function, eigenvalue } -(\tfrac{1}{4}+t^2))$$

For a left K-invariant function f on G/K with sufficient decay, the **spherical transform** $\widetilde{f}$ of f is

$$\widetilde{f}(\xi) \;=\; \int_G f\cdot\overline{\varphi}_{\frac{1}{2}+i\xi} \;=\; \int_G f\cdot\varphi_{\frac{1}{2}+i\xi} \qquad\qquad \text{(with real } \xi)$$

Spherical **inversion** is

$$f \;=\; \frac{8}{\pi}\int_{-\infty}^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\varphi_{\frac{1}{2}+i\xi}\cdot\xi^2\,d\xi$$

The Plancherel theorem for f, F left-and-right K-invariant functions in $L^2(G)$ is

$$\int_G f\cdot F \;=\; \frac{8}{\pi}\int_0^{\infty} \widetilde{f}(\tfrac{1}{2}+i\xi)\cdot\widetilde{F}(\tfrac{1}{2}+i\xi)\cdot\xi^{-2}\,d\xi$$

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2. *Formula for triple integrals of eigenfunctions*

The integral of three (or more) zonal spherical functions for $SL_2(\mathbb{C})$ can be expressed in elementary terms. This section carries out the computation for three. In fact, the expressions produced are not as useful as one might have anticipated, because extraction of useful asymptotics is not trivial. Direct determination of asymptotics for arbitrary products is done subsequently.

Since $\varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}$ is in $L^2(G)$ for $a, b \in \mathbb{R}$, this product has a reasonable spherical transform

$$\big(\varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}\big)^{\widetilde{}}(c) \;=\; \int_G \varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}\,\varphi_{\frac{1}{2}+ic}$$

$$=\; \frac{1}{8abc}\int_0^{\infty} \frac{\sin 2ar\;\sin 2br\;\sin 2cr}{\sinh^3 r}\;\sinh^2 r\;dr \;=\; \frac{1}{16abc}\int_{-\infty}^{\infty} \frac{\sin 2ar\;\sin 2br\;\sin 2cr}{\sinh r}\;dr$$

These integrals are expressible in terms of integrals

$$\int_{-\infty}^{\infty} \frac{e^{itr}\;\sin ur}{\sinh r}\,dr$$

The latter is holomorphic in t and u for t, u near the real axis. With $t > |u|$, by residues,

$$\int_{-\infty}^{\infty} \frac{e^{itr}\;\sin ur}{\sinh r}\;dr \;=\; 2\pi i\cdot\sum_{\ell=1}^{\infty}(-1)^{\ell}\,e^{it(\pi i\ell)}\;\sin u\pi i\ell \;=\; \pi\sum_{\ell=1}^{\infty}(-1)^{\ell}\,e^{it(\pi i\ell)}\,\big(e^{iu(\pi i\ell)} - e^{-iu(\pi i\ell)}\big)$$

$$=\; \pi\Big(\frac{-e^{-\pi(t+u)}}{1+e^{-\pi(t+u)}} - \frac{-e^{-\pi(t-u)}}{1+e^{-\pi(t-u)}}\Big) \;=\; \pi\,\frac{e^{-\pi(t-u)} - e^{-\pi(t+u)}}{(1+e^{-\pi(t+u)})\,(1+e^{-\pi(t-u)})} \;=\; \frac{\pi}{2}\,\frac{\sinh \pi u}{\cosh\frac{\pi}{2}(t+u)\;\cosh\frac{\pi}{2}(t-u)}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\sin 2ar\ \sin 2br\ \sin 2cr}{\sinh\ r}\, dr \;=\; -\frac{1}{4}\sum_{\pm,\pm}\int_{-\infty}^{\infty} \frac{e^{i(\pm 2a\pm 2b)r}\ \sin 2cr}{\sinh\ r}\, dr$$

$$= \frac{-\pi\sinh 2\pi c}{8}\Big(\frac{1}{\cosh\pi(a+b+c)\,\cosh\pi(a+b-c)} - \frac{1}{\cosh\pi(a-b+c)\,\cosh\pi(a-b-c)}$$
$$- \frac{1}{\cosh\pi(-a+b+c)\,\cosh\pi(-a+b-c)} + \frac{1}{\cosh\pi(-a-b+c)\,\cosh\pi(-a-b-c)}\Big)$$
$$= \frac{-\pi\sinh 2\pi c}{4}\Big(\frac{1}{\cosh\pi(a+b+c)\,\cosh\pi(a+b-c)} - \frac{1}{\cosh\pi(a-b+c)\,\cosh\pi(a-b-c)}\Big)$$

Toward putting these fractions over a common denominator, recall

$$\cosh(A-B+C)\,\cosh(A-B-C) \;-\; \cosh(A+B+C)\,\cosh(A+B-C)$$
$$= \frac{1}{4}\big(e^{2A-2B}+e^{2C}+e^{-2A+2B}+e^{-2C}\big) - \frac{1}{4}\big(e^{2A+2B}+e^{2C}+e^{-2C}+e^{-2A-2B}\big)$$
$$= -\sinh 2A\cdot\sinh 2B$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\sin 2ar\ \sin 2br\ \sin 2cr}{\sinh\ r}\, dr \;=\; \frac{\pi}{4}\cdot\frac{\sinh 2\pi a\ \sinh 2\pi b\ \sinh 2\pi c}{\cosh\pi(a+b+c)\cosh\pi(a+b-c)\cosh\pi(a-b+c)\cosh\pi(a-b-c)}$$

With the factor of $1/16abc$,

$$\int_G \varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}\,\varphi_{\frac{1}{2}+ic} \;=\; \frac{\pi}{64abc}\cdot\frac{\sinh 2\pi a\ \sinh 2\pi b\ \sinh 2\pi c}{\cosh\pi(a+b+c)\cosh\pi(a+b-c)\cosh\pi(a-b+c)\cosh\pi(a-b-c)}$$

[2.0.1] Remark: It is clear that this formula yields asymptotics as one or more of the parameters a, b, c becomes large. For b, c fixed and $a\to\infty$, exponential decay is clear. For c fixed while $a = b\to\infty$, the triple integral only decays like $1/ab$. For $a = b = c\to\infty$, again there is no exponential decay.

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3. *Asymptotics for triple integrals*

The integral of three zonal spherical functions has different behavior in different parameter regimes, readily seen from the explicit formula above. More economical methods scale better and give cleaner results, as follows. These can be viewed as elementary analogues of [Krötz-Stanton 2004], using analytic continuation to a thickening of the space in its complexification.

Without loss of generality, $a\ge b\ge c\ge 0$.

[3.1] The main case: asymptotics for $a > |b| + |c|$

First treat the case that a goes to ∞ faster than b, c. This scenario occurs in the spectral decomposition of a product of two eigenfunctions. Move the contour of integration in

$$I(a,b,c) \;=\; \int_{-\infty}^{\infty} \frac{e^{2iax}\ \sin 2bx\ \sin 2cx}{\sinh x}\, dx$$

upward by an amount $\pi < h < 2\pi$ across the first pole πi in the upper half-plane, producing a main term and an error integral:

$$I(a,b,c) \;=\; 2\pi i\, e^{-2\pi a}\ \sinh 2\pi b\ \sinh 2\pi c \;+\; \int_{-\infty}^{\infty} \frac{e^{2ia(x+ih)}\ \sin 2b(x+ih)\ \sin 2c(x+ih)}{\sinh(x+ih)}\, dx$$

From the identities

$$\sin(x+ih) \;=\; \sin x\cdot\cos ih+\sin ih\cdot\cos x \;=\; \sin x\cdot\cosh(-h)-i\sinh(-x)\cdot\cos x$$

and

$$\sinh(x+ih) \;=\; \sinh x\cdot\cosh ih+\sinh ih\cdot\cosh x \;=\; \sinh x\cdot\cos h+i\sin x\cdot\cosh x$$

for $a\geq 0$ the error integral is estimated by

$$\Big|\int_{-\infty}^{\infty}\frac{e^{2ia(x+ih)}\,\sin 2b(x+ih)\,\sin 2c(x+ih)}{\sinh(x+ih)}\,dx\Big| \;\ll_h\; e^{-2h\cdot(a-|b|-|c|)} \qquad\qquad (\text{with } a\geq 0)$$

The same idea applies to e^{-2iax}, moving the contour down rather than up, producing another copy of the main term and the same size error term. Thus, for every $\varepsilon>0$,

$$\int_{-\infty}^{\infty}\frac{\sin 2ax\,\sin 2bx\,\sin 2cx}{\sinh x}\,dx=\pi\,e^{-2\pi a}\,\sinh 2\pi b\,\sinh 2\pi c+O_\varepsilon(e^{-(4\pi-\varepsilon)(a-|b|-|c|)}) \qquad (\text{for } a\geq 0)$$

In the regime $a>|b|+|c|$ the error term is smaller than the main term, giving an asymptotic with error term: for every $\varepsilon>0$,

$$\int_G \varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}\,\varphi_{\frac{1}{2}+ic} \;=\; \frac{\pi e^{-2\pi a}\,\sinh 2\pi b\,\sinh 2\pi c}{16abc} \;+\; O_\varepsilon\Big(\frac{e^{-(4\pi-\varepsilon)(a-|b|-|c|)}}{16abc}\Big) \qquad (\text{for } a>|b|+|c|)$$

[3.2] Secondary case: asymptotics for $0\leq a<|b|+|c|$ and $a\geq b\geq c$

Now consider the secondary case that a does *not* go to infinity faster than the others, but, rather, $a<b+c$. Use the identity

$$\sin 2ax\cdot\sin 2bx \;=\; \tfrac{1}{2}\cos 2(a+b)x \;-\; \tfrac{1}{2}\cos 2(a-b)$$

to rewrite

$$\int_{-\infty}^{\infty}\frac{\sin 2ax\,\sin 2bx\,\sin 2cx}{\sinh x}\,dx \;=\; \int_{-\infty}^{\infty}\frac{\big(\frac{1}{2}\cos 2(a+b)x-\frac{1}{2}\cos 2(a-b)\big)\,\sin 2cx}{\sinh x}\,dx$$

Since $a+b>c\geq 0$, shifting contours as above, for all $\varepsilon>0$,

$$\int_{-\infty}^{\infty}\frac{\frac{1}{2}\cos 2(a+b)x\,\sin 2cx}{\sinh x}\,dx \;=\; \tfrac{1}{2}e^{-2\pi(a+b)}\,\sinh 2c+O_\varepsilon(e^{-(4\pi-\varepsilon)(a+b-c)}$$

But this will prove to be smaller than the eventual main term, and the error term absorbed entirely. Indeed, the assumption $a<b+c$ gives $a-b<c$, and the above argument, with c and $a-b$ now in the former roles of $a+b$ and c, gives

$$\int_{-\infty}^{\infty}\frac{\cos 2(a-b)x\,\sin 2cx}{\sinh x}\,dx \;=\; \tfrac{1}{2}e^{-2\pi c}\cosh 2\pi(a-b)+O_\varepsilon(e^{-(4\pi-\varepsilon)(c-|a-b|)} \qquad (\text{for } c>|a-b|)$$

Thus,

$$\int_{-\infty}^{\infty}\frac{\sin 2ax\,\sin 2bx\,\sin 2cx}{\sinh x}\,dx$$

$$=\;-\tfrac{1}{2}e^{-2\pi c}\cosh 2\pi(a-b)+O_\varepsilon(e^{-(4\pi-\varepsilon)(c-(a-b))}+\tfrac{1}{2}e^{-2\pi(a+b)}\,\sinh 2c+O_\varepsilon(e^{-(4\pi-\varepsilon)(a+b-c)}$$

Since $a\geq b\geq c\geq 0$, always $a+b-c\geq c-(a-b)$. Thus, one of the error terms is absorbed by the other, and

$$\int_{-\infty}^{\infty}\frac{\sin 2ax\,\sin 2bx\,\sin 2cx}{\sinh x}\,dx \;=\; -\tfrac{1}{2}e^{-2\pi c}\cosh 2\pi(a-b)+\tfrac{1}{2}e^{-2\pi(a+b)}\,\sinh 2c+O_\varepsilon(e^{-(4\pi-\varepsilon)(c-(a-b))}$$

The error term absorbs the term $e^{-2\pi(a+b)}\sinh 2c$ when the obvious comparison of exponents holds:

$$a+b-c \;\ge\; 2(c-(a-b))$$

which is

$$a \;\ge\; \frac{1}{3}b+c$$

Indeed, even where $a < b+c$, possibly $a \ge \frac{1}{3}b+c$. In fact, the *remaining* case

$$\begin{cases} a & < & b+c & & & & & \text{(negating } a \ge b+c\text{)} \\ a & < & \frac{1}{3}b+c & & & & & \text{(negating } a \ge \frac{1}{3}b+c\text{)} \\ a & \ge & b & \ge & c & \ge & 0 & \end{cases}$$

is atypical, in the sense that dehomogenizing the inequalities by dividing through by c yields a *bounded* region. The other two regions are unbounded even after dehomogenizing.

[3.3] Summary

Simply because $\varphi_{\frac{1}{2}+ib}\varphi_{\frac{1}{2}+ic}$ is *smooth* and has L^2 derivatives of all orders, the triple integral of $\varphi_{\frac{1}{2}+ia}\varphi_{\frac{1}{2}+ib}\varphi_{\frac{1}{2}+ic}$ is *rapidly* decreasing in a for fixed values of b, c. But this a weaker assertion than what is true, since this coefficient is *exponentially* decreasing. In fact, It has an asymptotic expansion with a strictly smaller exponential error term: for every $\varepsilon > 0$,

$$\int_G \varphi_{\frac{1}{2}+ia}\,\varphi_{\frac{1}{2}+ib}\,\varphi_{\frac{1}{2}+ic} \;=\; \frac{\pi e^{-2\pi a}\,\sinh 2\pi b\,\sinh 2\pi c}{16abc} \;+\; O_\varepsilon\Big(\frac{e^{-(4\pi-\varepsilon)(a-|b|-|c|)}}{16abc}\Big) \qquad \text{(for } a \ge |b|+|c|\text{)}$$

On the other hand, scenarios in which the two largest parameters increase at about the same rate produce exponential decay in the *smallest* parameter. In particular, when the smallest parameter is bounded and the difference between the larger two parameters is bounded, there is definitely *not* decay.

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4. *Asymptotics of integrals of n-fold products*

An n-fold integral of zonal spherical functions

$$\int_G \varphi_{\frac{1}{2}+ic_1}\cdots\varphi_{\frac{1}{2}+ic_n} \;=\; \frac{1}{2^{n+1}c_1\ldots c_n}\int_{-\infty}^{\infty} \frac{\sin 2c_1x\ldots\sin 2c_nx}{\sinh^{n-2}x}\,dx$$

is the c_1^{th} spectral decomposition coefficient of the product of $n-1$ zonal spherical functions. We are first interested in its asymptotics as a function of c_1, with $c_2,\ldots,c_n$ fixed. Other parameter configurations are subtler.

[4.1] Main case: $c_1 > |c_2|+\ldots+|c_n|$

Without loss of generality, take

$$c_1 \;\ge\; c_2 \;\ge\; \ldots \;\ge\; c_1 \;\ge\; 0$$

Moving the contour upward by an amount $\pi < h < 2\pi$,

$$\int_{-\infty}^{\infty} \frac{e^{2ic_1x}\,\sin 2c_2x\ldots\sin 2c_nx}{\sinh^{n-2}x}\,dx$$

$$=\; 2\pi i\mathrm{Res}_{x=\pi i}\frac{e^{2ic_1x}\,\sin 2c_2x\ldots\sin 2c_nx}{\sinh^{n-2}x} \;+\; \int_{-\infty}^{\infty} \frac{e^{2ic_1(x+ih)}\,\sin 2c_2(x+ih)\ldots\sin 2c_n(x+ih)}{\sinh^{n-2}(x+ih)}\,dx$$

$$2\pi i \mathrm{Res}_{x=\pi i} \frac{e^{2ic_1x}\ \sin 2c_2x \ldots \sin 2c_nx}{\sinh^{n-2} x} \;+\; O_\varepsilon(c_1^{n-3}\, e^{-(4\pi-\varepsilon)(c_1-|c_2|-\ldots-|c_n|)})$$

where $h = 4\pi - \varepsilon$. The indicated residue is a finite sum of exponentials of the form

$$e^{-2\pi(c_1\pm c_2\pm c_3\pm\ldots\pm c_n)}$$

with polynomial coefficients of degree at most $n-3$. Thus, the indicated error is much smaller than the main terms, and a precise asymptotic follows. In particular, with $c_2,\ldots,c_n$ fixed,

$$\int_G \varphi_{\frac{1}{2}+ic_1}\cdots\varphi_{\frac{1}{2}+ic_n} \;\ll\; |c_1|^{n-4}\, e^{-2\pi c_1} \qquad\qquad (\text{with } c_2,\ldots,c_n \text{ fixed})$$

[4.2] An opposite case: two largest parameters close together

Continue to take $c_1 \geq \ldots \geq c_n \geq 0$, without loss of generality. Now assume $c_1 - c_2$ is small. In particular, suppose that

$$c_3 \;\geq\; |c_4| + \ldots + |c_n| + |c_1 - c_2|$$

This requires something of the parameters $c_4, c_5, \ldots$ when $n \geq 4$. Rewrite the n-fold integral as a difference of two similar integrals using

$$\sin 2c_1x \cdot \sin 2c_2x \;=\; \tfrac{1}{2}\cos 2(c_1+c_2)x \;-\; \tfrac{1}{2}\cos 2(c_1-c_2)x$$

Since certainly $c_1+c_2 \geq c_3+\ldots+c_n$, the contour-shifting argument shows that the integral with $\cos 2(c_1+c_2)x$ is dominated by

$$(c_1+c_2)^{n-3}\, e^{-(4\pi-\varepsilon)(c_1+c_2-|c_3|-\ldots-|c_n|)}$$

This error will be absorbed easily by the error term in estimating the integral with $\cos 2(c_1-c_2)x$. For the latter, use the fact that

$$c_3 \;>\; |c_4| + \ldots + |c_n| + |c_1 - c_2|$$

without concern for the size of $c_1 - c_2$ relative to $c_3,\ldots,c_n$. The contour-shifting argument proves that the corresponding integral is a finite sum of exponentials of the form

$$e^{-2\pi(c_3\pm|c_4|\pm\ldots\pm|c_n|\pm|c_1-c_2|)}$$

with polynomial coefficients of degree at most $n-3$, with an error on the order of

$$|c_3|^{n-3}\, e^{-(4\pi-\varepsilon)(c_3-|c_4|-\ldots-|c_n|-|c_1-c_2|)}$$

Thus, the first error term is absorbed by this error term. Then it is clear that the term (up to a non-zero constant)

$$(c_3 - |c_4| - \ldots - |c_n| - |c_1-c_2|)^{n-3}\, e^{-2\pi(c_3-|c_4|-\ldots-|c_n|-|c_1-c_2|)}$$

is largest. Thus, for $c_1 - c_2$ small in this sense, under various mild hypotheses the n-fold integral is *bounded away from* 0, or even asymptotic to a *non-zero constant.*

[4.2.1] Remark: An explicit elementary expression for the n-fold integral of zonal spherical harmonics can be obtained by the same devices as for the three-fold integral, but such an expressions seems sub-optimal for understanding the asymptotics of the decay.

[Berezin 1956a] F.A. Berezin, *Laplace operators on semisimple Lie groups*, Dokl. Akad. Nauk SSSR **107** (1956), 9-12.

[Berezin 1956b] F.A. Berezin, *Representations of complex semisimple Lie groups in Banach spaces*, Dokl. Akad. Nauk SSSR **110** (1956), 897-900.

[Bernstein-Reznikoff 1999] J. Bernstein, A. Reznikoff, *Analytic continuation of representations and estimates of automorphic forms*, Ann. of Math. **150** (1999), 329-352.

[Gangolli-Varadarajan 1988] R. Gangolli, V.S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups*, Springer-Verlag, 1988.

[Gelfand-Naimark 1950] I.M. Gelfand, M.A. Naimark, *Unitary representations of the classical groups*, Trudy Mat.Inst. Steklova **36** (1950), 1-288.

[Godement 1948] R. Godement, *A theory of spherical functions I*, Trans. AMS, **73** (1952), 496-536.

[Harish-Chandra 1951] Harish-Chandra, *Plancherel formula for complex semisimple Lie groups*, Proc. Nat. Acad. Sci. USA **37** (1951) 813-818.

[Harish-Chandra 1952] Harish-Chandra, *Plancherel formula for the* 2×2 *real unimodular group*, Proc. Nat. Acad. Sci. USA **38** (1952), 337-342.

[Harish-Chandra 1954] Harish-Chandra, *The Plancherel formula for complex semisimple Lie groups*, Trans. AMS **76** (1954), 485-528.

[Helgason 1984] S. Helgason, *Groups and geometric analysis*, Academic Press, 1984.

[Knapp 1986] A. Knapp, *Representation theory of semi-simple real Lie groups: an overview based on examples*, Princeton University Press, 1986.

[Krötz-Stanton 2004] B. Krötz, R. Stanton, *Holomorphic extension of representations: (I) automorphic functions*, Ann. of Math. **159** (2004), 641-724. arXiv:math.RT/0210111.

[Sarnak 1994] P. Sarnak, *Integrals of products of eigenfunctions*, IMRN **6** (1994), 251-260.

[Stanton-Tomas 1978] R.J. Stanton, P.A. Tomas, *Expansions for spherical functions on non-compact symmetric spaces*, Acta Math. **140**, 251-276.

[Varadarajan 1989] V. S. Varadarajan, *An introduction to harmonic analysis on semisimple Lie groups*, Cambridge University Press, 1989.