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An unlikely distribution

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First, a well-known example: the principal-value integral

$$\eta(f) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{f(x)}{x}\,dx \qquad\qquad \text{(test functions } f\text{)}$$

is a *distribution*, and satisfies $x\cdot\eta = 1$: for test functions f,

$$(x\cdot\eta)(f) \;=\; \eta(x\cdot f) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{x\cdot f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} f(x)\,dx \;=\; \int_{\mathbb{R}} 1\cdot f(x)\,dx \;=\; 1(f)$$

However, this does *not* imply (by the obvious symbolic division), that η is (integration-against) $1/x$. Among other obstacles, $1/x$ is not locally L^1. Nevertheless, again, $x\cdot\eta = 1$, as distributions.

Compatibly, the meromorphic family of *odd* distributions $\mathrm{sgn}(x)/|x|^s$ has a meromorphic continuation to $s\in\mathbb{C}$, and is discovered to be holomorphic at $s=1$, for various reasons. From this, *some* incarnation of the pointwise function $\mathrm{sgn}(x)/|x| = 1/x$ gives a distribution, despite the pointwise function $1/x$ not being locally L^1. As checked above, the principal value integral against $1/x$ is one formulaic expression for this distribution.

In considerable parallel, the *even* function $v = 1/|x|$ is not locally L^1. It satisfies $x\cdot v = \mathrm{sgn}(x)$. Note that $\mathrm{sgn}(x)$ is the *odd* degree-zero distribution, 1 is the *even* degree-zero distribution, and

$$x\cdot 1/x \;=\; 1 \qquad\qquad x\cdot 1/|x| \;=\; \mathrm{sgn}(x) \qquad\qquad \text{(as pointwise functions)}$$

However, in contrast to the *odd* case, the corresponding family of *even* distributions, $1/|x|^s$, *does* have a pole at $s=1$, and the residue is (a constant multiple of) δ. Further:

[0.1] Claim: The integrate-against-$1/|x|$ functional does *not* extend from the subspace of test functions vanishing at 0 to an even, homogeneous distribution.

Proof: Suppose it did extend to such a functional u. Let f be a test function. For $t>0$, by homogeneity,

$$0 \;=\; u(f) - u(f\circ t) \;=\; u(f - f\circ t)$$

Since $f - f\circ t$ vanishes at 0,

$$u(f - f\circ t) \;=\; \int_{\mathbb{R}} \frac{1}{|x|}\big(f(x) - f(tx)\big)\,dx$$

Let f be positive and monotone decreasing going away from 0, so that the derivative is positive to the left and negative to the right. For $t>1$ and all $x\neq 0$, $f(x) - f(tx) > 0$, so

$$0 \;=\; \int_{\mathbb{R}} \frac{1}{|x|}\big(f(x) - f(tx)\big)\,dx \;>\; 0$$

contradiction. ///

Thus, although $x\cdot 1/|x| = \mathrm{sgn}(x)$ as a pointwise function away from 0, $1/|x|$ does not extend to a homogeneous, even distribution. (Of course, Hahn-Banach or more direct constructions can make *some* extension, but it will not be homogeneous and even.)

Nevertheless, we claim that there is an even (tempered) distribution v such that $x\cdot v = \mathrm{sgn}$. This would *seem* to suggest that (in some regularized sense) $v = \mathrm{sgn}/x = 1/|x|$, which would contradict the demonstrated non-existence of a reasonable regularization of $1/|x|$. Also, the equation $x \cdot v = \mathrm{sgn}$ might suggest that v is homogeneous (at least up to some multiple of δ, which is annihilated by multiplication by x).

[0.2] Claim: (Up to constants) the Fourier transform $\widehat{u}$ of $u = \log|x|$ satisfies $x \cdot \widehat{u} = \mathrm{sgn}$.

Proof: First, $\partial u/\partial x = \eta$. Taking Fourier transform, up to constants,

$$x \cdot \widehat{u} \;=\; \widehat{\eta} \;=\; \mathrm{sgn}$$

by computing one way or another that the Fourier transform of η is a multiple of sgn. ///

[0.3] To distinguish the two cases, note that 1 and sgn are the two degree-zero positive-homogeneous distributions, even and odd, respectively. Apply $\partial = \partial/\partial x$. With $x \cdot v = 1$, this gives $\partial x v = 0$, or $(x\partial + 1)v = 0$. In contrast, $x \cdot v = \mathrm{sgn}$ gives $(x\partial + 1)v = 2\delta \neq 0$.

The differential equation $(x\partial - s)u = 0$ is the equation for positive-homogeneity of degree s, so a solution to $(x\partial + 1)v = 0$ is positive-homogeneous of degree -1, as we know the principle-value integral of $1/x$ to be.

The differential equation $(x\partial + 1)u = 2\delta$ is obviously not a homogeneous equation, so a solution u is not positive-homogeneous of degree -1. Still, δ does satisfy $(x\partial + 1)\delta = 0$, as expected from its positive-homogeneity of degree -1. In particular, $(x\partial + 1)^2 u = 0$. That is, u is a *generalized eigenvector* of the Euler operator $x\partial$.

[0.4] Origin of $\log|x|$ is in *variation of parameters*, which produces generalized eigenvectors from a parametrized family of eigenvectors. Here, the operator is the Euler operator $x\partial$, with addition constraints of parity. The case of immediate interest is *even* eigenfunctions and generalized eigenfunctions.

The meromorphic family $|x|^s$ with $s \in \mathbb{C}$ satisfies the eigenfunction equation(s) $(x\partial - s)|x|^s = 0$. Differentiating with respect to s gives

$$-|x|^s + (x\partial - s)(\log|x| \cdot |x|^s) \;=\; 0$$

or

$$(x\partial - s)(\log|x| \cdot |x|^s) \;=\; |x|^s$$

Thus, $(x\partial - s)^2(\log|x| \cdot |x|^s) = 0$. Evaluating at $s = 0$, $(x\partial)\log|x| = 1$. As above, taking Fourier transform, $-\partial x\,\widehat{\log|x|} = \delta$, which gives

$$(x\partial + 1)\widehat{\log|x|} \;=\; -\delta$$
