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Volume of $SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})$ *and* $Sp_n(\mathbb{Z})\backslash Sp_n(\mathbb{R})$

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[Minor edits to the Feb 19, 2005 version.]

We follow Siegel to prove by induction that, up to elementary normalizations,

$$\mathrm{vol}\,(SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})) \;=\; \zeta(2)\,\zeta(3)\,\zeta(4)\,\zeta(5)\ldots\zeta(n)$$

Mysterious $\zeta(\text{odd})$ values appear. In contrast, for symplectic groups

$$\mathrm{vol}\,(Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{R})) \;=\; \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

In particular, for symplectic groups the values of zeta at odd integers do *not* appear.

In both cases, Poisson summation plays a critical role. To express volumes of other classical groups, Poisson summation must be replaced by subtler devices.

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1. *Volume of* $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$

Let $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z})$. To describe a right G-invariant measure on $\Gamma\backslash G$, it suffices to tell how to integrate compactly-supported continuous functions on $\Gamma\backslash G$. One first proves that, given a compactly-supported continuous function F on $\Gamma\backslash G$, there is a compactly-supported continuous function f on G so that

$$F(g) \;=\; \sum_{\gamma\in\Gamma} f(\gamma\cdot g)$$

Then define

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_G f(g)\,dg$$

(and verify that this is well-defined, meaning that it is independent of the choice of f).

To describe the measure on G, let K be the usual special orthogonal group

$$K \;=\; SO(2) \;=\; \{g\in G : g^\top g = 1_2\}$$

and let P be the standard parabolic subgroup

$$P \;=\; \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\}$$

with subgroup

$$P^+ \;=\; \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a>0,\ b\in\mathbb{R}\}$$

Recall the Iwasawa decomposition

$$G \;=\; P^+\cdot K \;\approx\; P^+\times K$$

The normalization of the Haar measure on G can be chosen so that for any absolutely integrable function φ on G

$$\int_G \varphi(g)\,dg \;=\; \int_{P^+}\int_K \varphi(pk)\,dk\,dp$$

where the Haar measure on K gives it total measure 2π, and where the *left* Haar measure dp on P^+ is normalized as follows. Let $p = na$ where

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and take

$$dp \;=\; d(na) \;=\; \frac{dn\,da}{t^2}$$

Choose an auxiliary Schwartz function f on $\mathbb{R}^2$ and define a function F on G by

$$F(g) \;=\; \sum_{v\in\mathbb{Z}^2} f(vg)$$

By design, this function F is left Γ-invariant. By evaluating

$$\int_{\Gamma\backslash G} F(g)\,dg$$

in two different ways we will determine the volume of $\Gamma\backslash G$.

For a fixed positive integer ℓ, the set $\{(c,d) : \gcd(c,d) = \ell\}$ is an orbit of Γ in $\mathbb{Z}^2$. We choose $(0,1)$ as a convenient base point and observe that

$$\mathbb{Z}^2 - \{0\} \;=\; \{\ell\cdot(0,1)\cdot\gamma :\; \gamma\in\Gamma, \ell > 0\}$$

The stabilizer of $(0,1)$ in Γ is

$$N_{\mathbb{Z}} \;=\; \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x\in\mathbb{Z}\}$$

so we have a bijection

$$\mathbb{Z}^2 - \{0\} \;\longleftrightarrow\; \{\ell > 0\}\times N_{\mathbb{Z}}\backslash\Gamma$$

given by

$$\ell\cdot(0,1)\gamma \;\leftarrow\; \ell\times N_{\mathbb{Z}}\gamma$$

Let

$$N \;=\; \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x\in\mathbb{R}\}\subset G$$

By unwinding the iterated integral

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_{\Gamma\backslash G} f(0)\,dg + \int_{\Gamma\backslash G}\sum_{v\neq 0} f(vg)\,dg \;=\; \int_{\Gamma\backslash G} f(0)\,dg + \sum_{\ell>0}\int_{N_{\mathbb{Z}}\backslash G} f(\ell\cdot(0,1)g)\,dg$$

where $N_{\mathbb{Z}} = N\cap\Gamma = P^+\cap\Gamma$. By expressing the Haar integral on G in terms of an iterated integral on P^+ and K

$$\int_{\Gamma\backslash G} f(0)\,dg + \sum_{\ell>0}\int_{N_{\mathbb{Z}}\backslash P}\int_K f(\ell\cdot(0,1)pk)\,dg$$

We choose the function f on $\mathbb{R}^2$ to be rotation invariant. Then

$$f(\ell(0,1)pk) = f(\ell(0,1)p)$$

and the integral becomes

$$\int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_{\ell>0}\int_{N_{\mathbb{Z}}\backslash P} f(\ell(0,1)p)\,dp$$

since the total measure of K is 2π. Write the Haar measure on P in terms of $N = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$ and $M = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\}$, to obtain

$$\int_{\Gamma\backslash G} f(0)\,dg + 2\pi\sum_{\ell}\int_M\int_{N_{\mathbb{Z}}\backslash N} f(\ell(0,1)nm)\,dn\,t^{-2}\,dm$$

where $m = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Note that

$$f(\ell(0,1)nm) \;=\; f(\ell(0,1)m)$$

so the integral over $n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ in N is just

$$\int_{N_{\mathbb{Z}}\backslash N} 1\,dx = \int_{\mathbb{R}/\mathbb{Z}} 1\,dx = 1$$

Thus, the whole integral is

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_{\ell}\int_M f(\ell(0,1)m)\,\frac{dm}{t^2} \;=\; \int_{\Gamma\backslash G} f(0)\,dg + 2\pi\cdot\sum_{\ell}\int_0^\infty f(\ell(0,t^{-1}))\,t^{-2}\,\frac{dt}{t}$$

$$=\; f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\cdot\sum_{\ell}\int_0^\infty f(0,\ell t)\,t^2\,\frac{dt}{t}$$

upon replacing t by t^{-1}. Replacing t by t/ℓ gives

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\cdot\sum_{\ell}\ell^{-2}\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t} \;=\; f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + 2\pi\,\zeta(2)\cdot\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t}$$

Further, using again the rotation invariance of f,

$$\int_0^\infty f(0,t)\,t^2\,\frac{dt}{t} \;=\; \int_0^\infty f(0,t)\,t\,dt \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)\,dx = \frac{1}{2\pi}\,\hat{f}(0)$$

Thus, the factors of 2π cancel, and altogether

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_{\Gamma\backslash G}\sum_{x\in\mathbb{Z}^2} f(xg)\,dg \;=\; f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0)$$

On the other hand, via Poisson summation,

$$\sum_{v\in\mathbb{Z}^2} f(vg) \;=\; \frac{1}{|\det g|}\sum_{v\in\mathbb{Z}^2}\hat{f}(v\,{}^\top g^{-1}) \;=\; \sum_{v\in\mathbb{Z}^2}\hat{f}(v\,{}^\top g^{-1})$$

(since $\det g = 1$). The group Γ is stable under transpose-inverse, so we can do a completely analogous computation with the roles of f and $\hat{f}$ reversed, finally obtaining

$$f(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,\hat{f}(0) \;=\; \int_{\Gamma\backslash G} F(g)\,dg \;=\; \hat{f}(0)\cdot\mathrm{vol}\,(\Gamma\backslash G) + \zeta(2)\,f(0)$$

from which follows

$$(f(0) - \hat{f}(0)) \cdot \mathrm{vol}\,(\Gamma\backslash G) \;=\; (f(0) - \hat{f}(0)) \cdot \zeta(2)$$

Take f such that $f(0) \neq \hat{f}(0)$ to obtain

$$\mathrm{vol}\,(\Gamma\backslash G) \;=\; \zeta(2)$$

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2. *Comparison with* $SL(2,\mathbb{Z})\backslash\mathfrak{H}$

We now reconcile the previous computation with the computation, in a somewhat different normalization, of the volume of $SL(2,\mathbb{Z})\backslash\mathfrak{H}$, where $\mathfrak{H}$ is the upper half-plane with the usual linear fractional transformation action of $SL(2,\mathbb{R})$. Integrating the traditional measure $dx\,dy/y^2$ on the usual fundamental domain

$$\mathbb{F} \;=\; \{z = x + iy \in \mathfrak{H} : |x| \leq \frac{1}{2} \;\; |z| \geq 1\}$$

one obtains $\pi/3$. It is worthwhile to see that this value is compatible with the group-theoretic value $\zeta(2) = \pi^2/6$ obtained above.

First, $\mathfrak{H} \approx G/K$ by $g(i) \leftarrow g$, since K is the isotropy group of the point $i \in \mathfrak{H}$. But at the same time the center $\{\pm 1_2\}$ of G, which also lies inside K, acts trivially on $\mathfrak{H}$. This effectively gives $\{\pm 1\}\backslash K$ total measure 1, thus giving K total measure 2, rather than 2π.

Second, the usual coordinates $z = x + iy$ on $\mathfrak{H}$ correspond to coordinates

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}$$

rather than

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$$

as above. The change of coordinates has the effect of doubling the measure in the y-coordinate by comparison to the t-coordinate.

Thus, based on the $\Gamma\backslash G$ computation above, we would expect the measure of $\mathbb{F}$ to be

$$\mathrm{vol}\,(\Gamma\backslash G) \times \frac{2}{2\pi} \times 2 = \frac{\pi}{3} = \frac{\pi^2}{6} \times \frac{2}{2\pi} \times 2 = \frac{\pi}{3}$$

This does match the direct computation in the $z = x + iy$ coordinates.

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3. *Volume of* $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})$ *by induction*

Now we prove by induction that, reasonably normalized,

$$\mathrm{vol}\,(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \zeta(2)\zeta(3)\zeta(4)\zeta(5)\ldots\zeta(n)$$

The normalization of measure needs explanation. First, let $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$. Given a compactly-supported continuous function F on $\Gamma\backslash G$, there is a compactly-supported continuous function f on G so that

$$F(g) \;=\; \sum_{\gamma\in\Gamma} f(\gamma \cdot g)$$

Then define

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_G f(g)\,dg$$

This is well-defined, meaning that it is independent of the choice of f. Thus, a right G-invariant measure on the quotient $\Gamma\backslash G$ is completely specified by choice of a Haar measure on G.

We reduce the normalization of a Haar measure on G to measures on subgroups. Let $K = SO(n)$, and let P^+ be the collection of upper-triangular real matrices with positive diagonal entries. Then $K \cap P^+ = 1_n$ and by the Iwasawa decomposition $G = P^+ \cdot K$. For a choice of Haar measure on K and choice of *left* Haar measure on P^+, for f compactly supported and continuous on G, the integral

$$f \to \int_{P^+}\int_K f(pk)\,dk\,dp$$

is a Haar integral on G. The normalization of the left Haar measure on P^+ is completely elementary, given by

$$d\begin{pmatrix} p_{11} & p_{12} & \ldots & & p_{1n} \\ 0 & p_{22} & \ldots & & \\ & & \ddots & & \\ & & & p_{n-1,n-1} & \\ & & & & \frac{1}{p_{11}p_{22}\cdots p_{n-1,n-1}} \end{pmatrix} = \prod_{1\le i<n} p_{ii}^{i+1-2n} \cdot \prod_{1\le i<n} \frac{dp_{ii}}{p_{ii}} \cdot \prod_{i<j} dp_{ij}$$

where the leading factor is the modular function on P^+.

Let f be a Schwartz function on $\mathbb{R}^n$ and define a function F on G by

$$F(g) \;=\; \sum_{v\in\mathbb{Z}^n} f(vg)$$

This function is left Γ-invariant. Consider

$$\int_{\Gamma\backslash G} F(g)\,dg$$

Let

$$Q \;=\; \{\begin{pmatrix} h & * \\ 0 & 1 \end{pmatrix} :\; h \in SL_{n-1}(\mathbb{R})\}$$

be the subgroup of G fixing $e = (0, 0, \ldots, 0, 1)$ under right multiplication. By linear algebra over $\mathbb{Z}$,

$$\mathbb{Z}^n - \{0\} = \sum_{\ell>0}\;\sum_{\gamma\in Q_\mathbb{Z}\backslash\Gamma} \ell\cdot e\cdot\gamma$$

where ℓ ranges over positive integers and $Q_\mathbb{Z} = Q \cap \Gamma$. Then

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_{\Gamma\backslash G} f(0)\,dg + \sum_\ell \int_{\Gamma\backslash G}\;\sum_{\gamma\in(Q_\mathbb{Z}\backslash\Gamma)} f(\ell e\gamma g)\,dg$$

where $Q_\mathbb{Z} = Q \cap \Gamma$. By unwinding, this is

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \sum_\ell \int_{Q_\mathbb{Z}\backslash G} f(\ell e g)\,dg$$

Let

$$P^+ \;=\; \{\begin{pmatrix} h & * \\ 0 & \frac{1}{\det h} \end{pmatrix} :\; \det h > 0\}$$

$$M \;=\; \{\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} :\; h \in SL_{n-1}(\mathbb{R})\}$$

$$A^+ \;=\; \{\begin{pmatrix} t^{\frac{1}{n-1}} \cdot 1_{n-1} & 0 \\ 0 & t^{-1} \end{pmatrix} :\; t > 0\}$$

$$N \;=\; \{\begin{pmatrix} 1_{n-1} & v \\ 0 & 1 \end{pmatrix} :\; v \in \mathbb{R}^{n-1}\}$$

Then $P^+ = NMA^+$ and $Q = NM$. Let $N_\mathbb{Z} = N \cap \Gamma$ and $M_\mathbb{Z} = M \cap \Gamma$. Via the Iwasawa decomposition $G = P^+ \cdot K$, and using induction, normalize the invariant integral so that for left-invariant functions Φ

$$\int_{Q_\mathbb{Z}\backslash G} \Phi(g)\, dg \;=\; \mathrm{vol}\,(S^{n-1}) \cdot \int_{A^+} \int_{Q_\mathbb{Z}\backslash NM} \int_K \Phi(nmak)\, t^{-n}\, dk\, dn\, dm\, da$$

where $\mathrm{vol}\,(S^{n-1})$ is the natural measure of the $(n-1)$-sphere S^{n-1}, and where

$$a \;=\; \begin{pmatrix} t^{\frac{1}{n-1}} \cdot 1_{n-1} & 0 \\ 0 & t^{-1} \end{pmatrix} :\; t > 0\}$$

The measure given by $t^{-n} dn\, dm\, da$ is a left Haar measure on P^+. The integral becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \sum_\ell \int_{A^+} \int_{Q_\mathbb{Z}\backslash NM} \int_K f(\ell \cdot e \cdot nmak)\, t^{-n}\, dk\, dn\, dm\, da$$

The integrand is invariant under NM, and the volume of $N_\mathbb{Z}\backslash N_\mathbb{R}$ is 1, so the whole becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \sum_\ell \int_{A^+} \int_K f(\ell \cdot e \cdot ak)\, t^{-n}\, dk\, da$$

For f right K-invariant this becomes

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \sum_\ell \int_{A^+} f(\ell e a)\, t^{-n}\, da$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \sum_\ell \int_0^\infty f(\ell e t^{-1})\, t^{-n}\, \frac{dt}{t}$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(S^{n-1}) \cdot \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \sum_\ell \frac{1}{\ell^n} \int_0^\infty f(et)\, t^n\, \frac{dt}{t}$$

upon replacing t by t^{-1}. Using the rotation-invariance of f,

$$\mathrm{vol}\,(S^{n-1}) \cdot \int_0^\infty f(et)\, t^n\, \frac{dt}{t} \;=\; \int_{\mathbb{R}^n} f(x)\, dx = \hat{f}(0)$$

Altogether,

$$\int_{\Gamma\backslash G} F(g)\, dg \;=\; \mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \zeta(n) \cdot \hat{f}(0)$$

By Poisson summation,

$$F(g) \;=\; \sum_{v \in \mathbb{Z}^n} f(vg) \;=\; \sum_{v \in \mathbb{Z}^n} \hat{f}(v\, {}^\top g^{-1}) \;=\; F({}^\top g^{-1})$$

The automorphism $g \to^\top g^{-1}$ preserves measure on G and stabilizes Γ. Since $f^{\hat{}\hat{}}(0) = f(0)$,

$$\mathrm{vol}\,(\Gamma\backslash G) f(0) + \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \zeta(n) \cdot \hat{f}(0) \;=\; \int_{\Gamma\backslash G} F(g)\, dg$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\hat{f}(0) + \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \zeta(n) \cdot f(0)$$

Taking f such that $f(0) \neq \hat{f}(0)$,

$$\mathrm{vol}\,(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) = \mathrm{vol}\,(SL_{n-1}(\mathbb{Z})\backslash SL_{n-1}(\mathbb{R})) \cdot \zeta(n)$$

By induction,

$$\mathrm{vol}\,(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R})) \;=\; \zeta(2)\zeta(3)\zeta(4)\zeta(5)\ldots\zeta(n)$$

The normalizations of measure appearing in the induction match the normalization described at the outset. For the contribution of K this is by design. The contribution of the subgroups P^+ is less clear. The normalization effectively given by the induction would put coordinates on diagonal matrices of determinant 1 by

$$\begin{pmatrix} t_1 t_2^{\frac{1}{2}} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} &&&&&& \\ & t_1^{-1} t_2^{\frac{1}{2}} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} &&&&& \\ && t_2^{-1} t_3^{\frac{1}{3}} \ldots t_{n-1}^{\frac{1}{n-1}} &&&& \\ &&& t_3^{-1} t_4^{\frac{1}{4}} \ldots t_{n-1}^{\frac{1}{n-1}} &&& \\ &&&& t_4^{-1} \ldots t_{n-1}^{\frac{1}{n-1}} && \\ &&&&& \ddots & \\ &&&&&& t_{n-1}^{-1} \end{pmatrix}$$

versus the coordinates

$$\begin{pmatrix} t_1 &&&& \\ & t_2 &&& \\ && \ddots && \\ &&& t_{n-1} & \\ &&&& \frac{1}{p_{11}p_{22}\cdots p_{n-1,n-1}} \end{pmatrix}$$

The lower right $(n-1)$-by-$(n-1)$ minor of the former has exponents

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{n-1} \\ 0 & -1 & \frac{1}{3} & \ldots & \frac{1}{n-1} \\ 0 & 0 & -1 & \ldots & \frac{1}{n-1} \\ &&& \ddots & \\ &&&& -1 \end{pmatrix}$$

which has determinant ± 1, so the change-of-measure going from one set of coordinates to the other is trivial. Thus, the measure used in the induction match the measure described at the beginning. ///

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4. *Symplectic groups*

Let $G = Sp(n, \mathbb{R})$ be the usual symplectic group of $2n$-by-$2n$ matrices, and $\Gamma = Sp(n, \mathbb{Z})$. With reasonably normalized measure,

$$\mathrm{vol}\,(Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{R})) = \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

The measure on $\Gamma\backslash G$ is determined from a Haar measure on G by the requirement that

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi(\gamma\cdot g)\,dg \;=\; \int_G \varphi(g)\,dg$$

for compactly-supported continuous φ on G. To specify a Haar measure on G, let

$$K \;=\; \{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A+iB\in U(n)\} \approx U(n)$$

with the usual unitary group

$$U(n) \;=\; \{h\in GL(n,\mathbb{C}) : h^*h = 1_n\}$$

where h^* is h-conjugate-transpose. And let P^+ be the subgroup of $Sp_n(\mathbb{R})$ consisting of elements of the form

$$\begin{pmatrix} t_1 & * & * & \ldots & * & \ldots & & * \\ 0 & t_2 & * & \ldots & \vdots & & & \\ \vdots & & \ddots & & & & & \vdots \\ 0 & \ldots & 0 & t_n & * & & \ldots & * \\ 0 & \ldots & & 0 & t_1^{-1} & 0 & \ldots & 0 \\ \vdots & & & & * & t_2^{-1} & & \vdots \\ & & & \vdots & \vdots & & \ddots & 0 \\ 0 & & \ldots & 0 & * & & & t_n^{-1} \end{pmatrix} = \begin{pmatrix} A & * \\ 0 & {}^\top A^{-1} \end{pmatrix} \quad (A \text{ upper-triangular})$$

Let N be the unipotent radical of P^+ (consisting of unipotent matrices in P^+). In these coordinates, a left Haar measure on P^+ is

$$t_1^{-2n}\,t_2^{-2n+2}\ldots t_{n-1}^{2n-2}\,t_n^{2n}\,dn\,\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n}$$

where dn is a Haar measure on N. Give K the Haar measure so that it has total measure

$$\mathrm{vol}\,(S^1)\mathrm{vol}\,(S^3)\mathrm{vol}\,(S^5)\ldots\mathrm{vol}\,(S^{2n-3})\mathrm{vol}\,(S^{2n-1})$$

where $\mathrm{vol}\,(S^k)$ is the standard volume of the k-sphere in $\mathbb{R}^{k+1}$. Then

$$\varphi \to \int_{P^+}\int_K f(pk)\,dp\,dk$$

is a Haar integral on G.

Let f be a Schwartz function on $\mathbb{R}^{2n}$, and define

$$F(g) = \sum_{v\in\mathbb{Z}^{2n}} f(v\cdot g)$$

viewing $v\in\mathbb{Z}^{2n}$ as a row vector. Evaluating $\int_{\Gamma\backslash G} F(g)\,dg$ in two different ways will allow evaluation of the volume of $\Gamma\backslash G$.

First, Γ is transitive on primitive elements in $\mathbb{Z}^{2n}$ (those whose entries have greatest common divisor 1), so

$$\mathbb{Z}^{2n} - \{0\} \;=\; \{\ell\cdot e\cdot\gamma :\; \ell>0,\; \gamma\in\Gamma\}$$

where

$$e \;=\; (\underbrace{0,\ldots,0}_{n},1,\underbrace{0,\ldots,0}_{n-1})$$

that is, with the lone 1 at the $(n+1)^{th}$ place. The isotropy group of e in G is

$$Q \;=\; \{\begin{pmatrix} 1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & 1 & 0 \\ 0 & c & * & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp_{n-1}(\mathbb{R})\}$$

and the other entries are of suitable sizes. Then

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \sum_{\ell>0} \int_{\Gamma\backslash G} \sum_{\gamma\in Q_{\mathbb{Z}}\backslash\Gamma} f(\ell\cdot e\cdot \gamma g)\,dg$$

Let

$$P^+ \;=\; \{\begin{pmatrix} t & * & * & * \\ 0 & a & * & b \\ 0 & 0 & t^{-1} & 0 \\ 0 & c & * & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n-1,\mathbb{R})\},\ t>0\}$$

$$M \;=\; \{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & t^{-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n-1,\mathbb{R}),\ t>0\}$$

$$A^+ \;=\; \{\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{pmatrix} \in P^+\} \qquad N \;=\; \{\begin{pmatrix} 1 & * & * & * \\ 0 & 1_{n-1} & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in Q\}$$

Then

$$P^+ \;=\; N\cdot M\cdot A^+ \qquad Q = N\cdot M$$

Note that $t^{-2n}\cdot dn\,dm\,da$ is a left Haar measure on P^+, with coordinate $a\in A^+$ as just above. Unwinding the integral, it is

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \sum_{\ell>0} \int_{Q_{\mathbb{Z}}\backslash G} f(\ell\cdot e\cdot g)\,dg$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot \sum_{\ell>0} \int_{A^+}\int_{M_{\mathbb{Z}}\backslash M}\int_{N_{\mathbb{Z}}\backslash N}\int_K f(\ell\cdot e\cdot nmak)\,t^{-2n}\;dk\,dm\,dn\,da$$

which for right K-invariant f is

$$\mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \sum_{\ell>0}\int_{A^+} f(\ell\cdot e\cdot a)\,t^{-2n}\,da$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \sum_{\ell>0}\int_0^\infty f(\ell\cdot t^{-1}\cdot e)\,t^{-2n}\,\frac{dt}{t}$$

Replacing t by t/ℓ gives

$$\mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(S^{2n-1})\cdot \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \zeta(2n)\int_0^\infty f(t\cdot e)\,t^{-2n}\,\frac{dt}{t}$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \zeta(2n)\int_{\mathbb{R}^{2n}} f(x)\,dx$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \zeta(2n)\,\hat{f}(0)$$

On the other hand, by Poisson summation

$$\int_{\Gamma\backslash G} F(g)\,dg \;=\; \int_{\Gamma\backslash G} \sum_{v\in\mathbb{Z}^{2n}} f(vg) \;=\; \int_{\Gamma\backslash G} \sum_{v\in\mathbb{Z}^{2n}} \hat{f}(v\,{}^\top g^{-1}) \;=\; \int_{\Gamma\backslash G} \sum_{v\in\mathbb{Z}^{2n}} \hat{f}(vg)$$

since the involution $g \to^\top g^{-1}$ preserves the Haar measure, and preserves Γ. Thus,

$$\mathrm{vol}\,(\Gamma\backslash G)\cdot f(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \zeta(2n)\,\hat{f}(0)$$

$$=\; \mathrm{vol}\,(\Gamma\backslash G)\cdot \hat{f}(0) + \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))\cdot \zeta(2n)\,f(0)$$

For f such that $f(0) \neq \hat{f}(0)$, solve for the volume

$$\mathrm{vol}\,(Sp_n(\mathbb{A})\backslash Sp_n(\mathbb{R})) \;=\; \zeta(2n)\cdot \mathrm{vol}\,(Sp_{n-1}(\mathbb{Z})\backslash Sp_{n-1}(\mathbb{R}))$$

Since $Sp(1) = SL(2)$, by induction, as claimed

$$\mathrm{vol}\,(Sp(n,\mathbb{Z})\backslash Sp(n,\mathbb{R})) \;=\; \zeta(2)\,\zeta(4)\,\zeta(6)\,\zeta(8)\ldots\zeta(2n)$$

Verification that the measure used in the induction agree with the measure specified at the outset is straightforward. ///