# von Neumann algebras

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*version August 15, 2000*

The point here is to define and describe some basic ideas about von Neumann algebras, without proofs, but including some relevant background material in compact form.

- Topologies on operators on Hilbert spaces
- Commutants, Schur's lemma, central characters
- The von Neumann density theorem
- Definition of von Neumann algebras, factor algebras
- Finite and infinite von Neumann algebras
- Basic classification of von Neumann algebras

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## Topologies on operators on Hilbert spaces

Let $V$ be a (complex) Hilbert space with inner product $\langle , \rangle$ and norm $||$. Let $\mathcal{B}(V)$ be the algebra of continuous linear operators $T : V \to V$. There are at least 3 important topologies on $\mathcal{B}(V)$.

The **uniform** or **norm** topology is the strongest topology we will consider, and gives $\mathcal{B}(V)$ the structure of *Banach space*. This topology is defined via the **operator norm**

$$|T| = \sup_{|v|=1} |Tv|$$

where as indicated $v$ ranges over unit vectors (in $V$).

The **strong topology** on $\mathcal{B}V$ is defined by a collection of *semi-norms*

$$\nu_v(T) = |Tv|$$

as $v$ ranges over $V$. Note that it is unlikely that there is a *countable* collection of semi-norms giving this topology, so it is therefore *not* obviously metrizable.

The **weak topology** on $\mathcal{B}V$ is defined by a collection of *semi-norms*

$$\nu_{v,w}(T) = |\langle Tv, w\rangle|$$

as $v, w$ range over $V$.

There are also ultra-strong and ultra-weak topologies, and others besides, but we don't need them here.

Let $\mathcal{B}(V)^\times$ be the group of continuous linear operators on a Hilbert space $V$ having continuous *inverses*.

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## Commutants, Schur's lemma

Let $A$ be a subalgebra of the algebra $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space $V$. Then the **commutant** $A'$ of $A$ is defined to be

$$A' = \{T \in \mathcal{B}(V) : T \circ \alpha = \alpha \circ T, \text{ for all } \alpha \in A\}$$

**Schur's Lemma** asserts that, if $(\pi, V)$ is an irreducible unitary Hilbert space representation of a topological group $G$, then the commutant $\rho(G)'$ of $\rho(G)$ consists just of the scalar operators $\mathbf{C} \cdot 1$ on $V$. This is an immediate consequence of elementary spectral theory for bounded operators on Hilbert spaces.

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# The von Neumann density theorem

Let $A$ be a subalgebra of the algebra $\mathcal{B}(V)$ of continuous linear operators on a Hilbert space $V$. Suppose that $A$ contains the scalar operators, and is stable under taking *adjoints*. Then von Neumann's **Density Theorem** asserts that

$$A'' = \text{ strong-topology closure of } A$$

This result is also called the **double commutant theorem**. It has analogues in other situations, as well.

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# Definition of von Neumann algebras, factor algebras

A **von Neumann algebra** $A$ is an adjoint-closed subalgebra of the algebra $\mathcal{B}(V)$ of bounded operators of a Hilbert space $V$, closed in the strong topology on operators.

Some sources require that a von Neumann algebra contain the scalar operators, or that it have a unit. We do not require the former condition, and note below that the latter property is *provable*.

In light of the density theorem, *if the scalar operators lie in the algebra*, then closedness in the strong operator topology is equivalent to the *purely algebraic* condition

$$A'' = A$$

Also, for any $*$-closed algebra $A$ in $\mathcal{B}(V)$ containing the scalars, since $A'$ is readily checked to be strong-topology closed, *the commutant $A'$ is a von Neumann algebra.*

A von Neumann algebra $A \subset \mathcal{B}(V)$ is a **factor** or **factor algebra** if

$$A \cap A' = \mathbf{C} \cdot 1$$

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# Finite and infinite von Neumann algebras

Perhaps surprisingly, *every non-zero von Neumann algebra has a unit* (although this unit certainly may not be the identity map in the algebra $\mathcal{B}(V)$ in which the von Neumann algebra lies).

A **projection** $p$ in $\mathcal{B}(V)$ is a self-adjoint element so that $p^2 = p$. The **rank** $rk(p)$ of a projection $p \in \mathcal{B}(V)$ is the dimension of its image (so is $\infty$ if not finite). Recall that a self-adjoint operator $T$ on $V$ is **positive**, written $T \geq 0$, if

$$\langle Tv, v \geq 0\rangle$$

for all $v \in V$. A projection $p$ is **finite** if for any other projection $q$ so that $p - q \geq 0$ and $rk(q) = rk(p)$ we have $p = q$. Otherwise, $p$ is said to be **infinite**. A projection $p$ in a von Neumann algebra $A$ is **abelian** if the subalgebra $pAp$ is abelian.

A von Neumann algebra is **finite** if its unit is a *finite* projection. A von Neumann algebra is **infinite** if its unit is an infinite projection.

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# Basic classification of von Neumann (factor) algebras

The classification of von Neumann algebras has a terminology involving *Types I,II,III* which is *not* directly related to the similar-sounding terminology for topological groups and $C^*$-algebras. At the level indicated here, these are really just definitions. The decomposition theory via Hilbert integrals is only slightly more substantial.

The classification is actually in terms of *factor algebras*, since one would prove the basic theorem that any von Neumann algebra can be decomposed as a Hilbert integral of factors.

A von Neumann algebra is
- **Type I** if every non-zero central projection in majorizes a non-zero *abelian* projection.
- **Type II** if is has *no* non-zero abelian projections and if every non-zero central projection majorizes a non-zero *finite* projection.
- **Type III** if it contains no non-zero finite projections.
- **properly infinite** if it has no non-zero finite central projections.
- **Type $\mathrm{II}_\infty$** if it is Type II and properly infinite.
- **Type $\mathrm{II}_1$** if it is Type II and finite.