APPLICATIONS OF A PRE–TRACE FORMULA TO ESTIMATES ON MAASS CUSP FORMS

ZHOU FANG

Abstract. By using spectral expansions in global automorphic Levi–Sobolev spaces, we estimate an average of the first Fourier coefficients of Maass cusp forms for $SL_2(\mathbb{Z})$, producing a soft estimate on the first numerical Fourier coefficients of Maass cusp forms, which is an example of a general technique for estimates on compact periods via application of a pre–trace formula. Incidentally, this shows that the distribution that evaluates the first Fourier coefficient of a Maass cusp form at height $y > 1$ lies in $-1/2 - \varepsilon$ global automorphic Levi–Sobolev space for every $\varepsilon > 0$. Moreover, we briefly explain the utility of Levi–Sobolev spaces and other modern analysis in the spectral theory of automorphic forms.

Contents

1. Review of Spectral Theory for $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$

1.1. **Maass Forms.** Let $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$ and $K = SO_2(\mathbb{R})$ as usual. By the Iwasawa decomposition, G/K can be identified with the upper half plane

$$\mathfrak{H} = \{x + iy \in \mathbb{C} : y > 0\},$$

on which Γ acts discontinuously by linear fractional transformations. A function $f : \mathfrak{H} \to \mathbb{C}$ is *automorphic*, provided $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathfrak{H}$. In addition, if f is an eigenfunction of

Date: October 6th, 2013. *Email*: fang0263@umn.edu.

the Laplace–Beltrami operator

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

and is of *moderate growth* (that is, $f(x+iy) \ll y^N$ for some positive integer N, as $y\to\infty$)[1], we call f a *Maass form*. We review the basic spectral theory of Maass form for $SL_2(\mathbb{Z})$ as follows (cf. [Sel56] [Roe56] [God66a] [God66b] [Hej83] [Iwa02] [Gar12a] [Gar11c]).

1.2. **Eisenstein Series.** The *Eisenstein series* arising in spectral theory is defined as

$$E_s(z) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma z)^s, \quad \text{for } \mathrm{Re}(s)>1,$$

in which P is the parabolic subgroup of G consisting of all upper–triangular matrices. By writing $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in P\cap\Gamma\backslash\Gamma$,

$$E_s(z) = \frac{1}{2}\sum_{\text{c,d coprime}} \frac{y^s}{|cz+d|^{2s}}.$$

We recall its basic properties as follows. E_s converges absolutely and uniformly in compact sets on $\mathfrak{H}$ for $\mathrm{Re}(s)>1$. In addition, it has a functional equation (determined by its constant term):

$$\xi(2s)E_s = \xi(2-2s)E_{1-s},$$

where $\xi(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is the completed zeta function with functional equation $\xi(s)=\xi(1-s)$. Moreover, $s(1-s)E_s$ admits an analytic continuation to an entire function of s. In $\mathrm{Re}(s)>\frac{1}{2}$, there is only one simple pole of E_s at $s=1$ with residue $\mathrm{Volume}(\Gamma\backslash\mathfrak{H})^{-1}=3/\pi$. In $0<\mathrm{Re}(s)<\frac{1}{2}$, E_s has poles at $s=s_0$ such that $2s_0$ are non–trivial zeros of the Riemann zeta function. In both its convergent and meromorphically continued regions, $E_s(x+iy)$ is of moderate growth in y for fixed s.

1.3. **Maass Cusp Forms.** *Pseudo–Eisenstein series* (or called *incomplete theta series*) is defined as

$$\Psi_\psi(z) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\psi(\mathrm{Im}(\gamma z)), \quad \text{for } \psi\in C_c^\infty(N\backslash G).$$

A Maass form f is a *Maass cusp form*, provided it is orthogonal to all pseudo–Eisenstein series. The 0^{th} Fourier coefficient (*constant term*) of a Maass cusp form is 0:

$$\int_{N\cap\Gamma\backslash N} f(ng)\,\mathrm{d}n = 0, \quad \text{for almost all } g\in G,$$

in which N is the subgroup of G consisting of upper–triangular *unipotent* matrices.

Remark 1.3.1. In the classical coordinate, the left G–invariant measure on $\mathfrak{H}$ is $\mathrm{d}x\,\mathrm{d}y/y^2$, with $\mathrm{d}x$ and $\mathrm{d}y$ the Lesbegue measures. In the Iwasawa decomposition $G=ANK$, the invariant Haar measure is given by $\mathrm{d}g=\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}k$.

In effect, a pseudo–Eisenstein series can be expressed as a contour integral of Eisenstein series on the critical line (up to a residual part) via Mellin transform:

$$\Psi_\psi = \frac{1}{2\pi i}\int_0^\infty \mathscr{M}\left(\text{the constant term of } \Psi_\psi\right)_{\frac{1}{2}+it}\cdot E_{\frac{1}{2}+it}\;\mathrm{d}t + \mathscr{M}\psi(1)\cdot \operatorname*{Res}_{s=1} E_s,$$

where $\mathscr{M}$ denotes the Mellin transform. However, it is important to note that $z\to E_s(z)$ does not lie in $L^2(\Gamma\backslash\mathfrak{H}) = \{L^2 \text{ Maass forms on } \mathfrak{H}\}$.

[1]In this note, P. G. H. Bachmann's "big O" and I. M. Vinogradov's "$\ll$" symbols are defined in the following way. $f=O(g)$ as $x\to x_0$ (or equivalently, $f\ll g$ for $x\in U$) precisely when there exists some neighborhood U of x_o and a constant M such that $f(x)\le M|g(x)|$ for $x\in U$.

1.4. L^2 **Automorphic Spectral Expansion.** Given $f \in L^2(\Gamma\backslash\mathfrak{H})$, its L^2 spectral expansion is

$$f = \sum_F \langle f, F\rangle F + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{-\infty}^{\infty} \left\langle f, E_{\frac{1}{2}+it}\right\rangle E_{\frac{1}{2}+it}\ \mathrm{d}t,$$

where F runs over an orthonormal basis of Maass cusp forms. Note that the integrals against Eisenstein series do not converge for all f in $L^2(\Gamma\backslash\mathfrak{H})$. The inner integral should be understood in terms of isometric extension from pseudo–Eisenstein series. Given a pseudo–Eisenstein series Ψ_ψ, the map

$$\mathcal{E}_1 : \{\text{pseudo–Eisenstein series}\} \longrightarrow \left\{u \in L^2\left(\frac{1}{2}+i\mathbb{R}\right) : u(-s) = \frac{\xi(2-2s)}{\xi(2s)}u(s),\ s \in \frac{1}{2}+i\mathbb{R}\right\}$$

given by

$$\mathcal{E}_1\left(\Psi_\psi\right) = \int_{\Gamma\backslash\mathfrak{H}} \Psi_\psi(z)E_{1-s}(z)\ \frac{\mathrm{d}x\ \mathrm{d}y}{y^2}$$

is an isometry (with dense image). We obatin the following isometry[2] (still denoted by $\mathcal{E}_1$) by extending $\mathcal{E}_1$ by continuity:

$$\mathcal{E}_1 : L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H})^\perp \longrightarrow \left\{u \in L^2\left(\frac{1}{2}+i\mathbb{R}\right) : u(-s) = \frac{\xi(2-2s)}{\xi(2s)}u(s),\ s \in \frac{1}{2}+i\mathbb{R}\right\},$$

in which $L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H}) = \{L^2 \text{ Maass cusp forms on } \mathfrak{H}\}$. Further, the outer integral should be understood in terms of isometric extension from test functions on the critical line via the Plancherel theorem on $\Gamma\backslash\mathfrak{H}$:

$$|f|^2_{L^2(\Gamma\backslash\mathfrak{H})} = \sum_F |\langle f, F\rangle|^2 + \frac{|\langle f, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{4\pi}\int_{-\infty}^{\infty} \left|\left\langle f, E_{\frac{1}{2}+it}\right\rangle\right|^2\ \mathrm{d}t.$$

Specifically, given $\widetilde{f} \in \left\{u \in C_c^\infty\left(\frac{1}{2}+i\mathbb{R}\right) : u(-s) = \frac{\xi(2-2s)}{\xi(2s)}u(s),\ s \in \frac{1}{2}+i\mathbb{R}\right\}$, the map (sometimes called Eisenstein transform)

$$\mathcal{E}_2 : \left\{u \in C_c^\infty\left(\frac{1}{2}+i\mathbb{R}\right) : u(-s) = \frac{\xi(2-2s)}{\xi(2s)}u(s),\ s \in \frac{1}{2}+i\mathbb{R}\right\} \longrightarrow L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H})^\perp$$

given by

$$\mathcal{E}_2(\widetilde{f}) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \widetilde{f}(t) \cdot E_{\frac{1}{2}+it}\ \mathrm{d}t$$

is an isometry. By the Plancherel theorem, we can extend $\mathcal{E}_2$ on test functions to the following isometry (still denoted by $\mathcal{E}_2$):

$$\mathcal{E}_2 : \left\{u \in L^2\left(\frac{1}{2}+i\mathbb{R}\right) : u(-s) = \frac{\xi(2-2s)}{\xi(2s)}u(s),\ s \in \frac{1}{2}+i\mathbb{R}\right\} \longrightarrow L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H})^\perp.$$

Remark 1.4.1. The notion of L^2 convergence is both appropriate and practical in our context. As has been known since Hecke, pointwise estimate will not be manageable here, since for instance, an optimal sup–norm estimate on $E_s(i) = \zeta_{\mathbb{Q}(i)}(s)/\zeta(2s)$ would prove the Lindelöf conjecture.

2. Averaged Estimate on the Fourier Coefficients of Maass Cusp Forms

Now suppose f is a Maass cusp form with eigenvalue $\lambda = s(s-1) \leq 0$. Since f is an *integer–translation invariant*[3] eigenfunction of Δ on $\mathfrak{H}$, its Fourier expansion is of the form $f(x+iy) = \sum_{m\neq 0} b_m(y,\lambda)e^{2\pi i x}$. Solving $(\Delta - \lambda)f = 0$ yields

$$f(x+iy) = \sum_{m\neq 0} c_m u(|m|y, \lambda)e^{2\pi i m x},$$

[2]In effect, this also proves the Plancherel theorem on the continuous spectrum.

[3]Since $f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} z\right) = f(z)$ by the automorphy of f, $f(z+1) = f(z)$.

with

$$u(y,\lambda) \sim e^{-2\pi y}\left(1+O\left(\frac{\lambda}{y}\right)\right), \text{ as } \lambda\to\infty \text{ uniformly for } y \text{ in compact subsets of } (1,\infty).^4$$

The actual size of c_1 remains mysterious. Though one may wish c_1 to be 1 so that Mellin transform would produce exactly the associated L–function, this does not occur. In the case of Hecke congruence subgroup of level N, for f a newform such that $|f|_{L^2(\Gamma\backslash\mathfrak{H})} = 1$, J. Hoffstein and P. Lockhart [HL94] established a highly non-trivial upper bound: $|c_1| \ll (-\lambda N)^{\varepsilon}$ for every $\varepsilon > 0$, which is based on a lower bound obtained earlier by H. Iwaniec [Iwa90]: $|c_1| \gg (-\lambda N)^{-\varepsilon}$.

Following [Gar10], we estimate an average of the first Fourier coefficients of Maass cusp forms and incidentally obtain a *soft* estimate on c_1 for $SL_2(\mathbb{Z})$ via application of a pre–trace formula.

Theorem 2.0.1. *Let $C_c^\infty(\Gamma\backslash\mathfrak{H})$ be the space of test functions inside $L^2_{cusp}(\Gamma\backslash\mathfrak{H})$. Given $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ with eigenvalue parameterized as $\lambda = s(s-1) \le 0$, let θ be the distribution that evaluates the first Fourier coefficient of f at height $y > 1$:*

$$\theta : f \longrightarrow \int_0^1 f(x+iy)e^{-2\pi i x}\, \mathrm{d}x.$$

We have

$$\sum_{F:|s_F|\ll T} |\theta(F)|^2 \ll_y T \qquad \textit{for all } T > 0,$$

where F runs over an orthonormal basis of Maass cusp forms.

Proof. It is convenient to first consider functions in $L^2(\Gamma\backslash G)$, and take K–invariant elements in the end. Let η be a left–and–right K–invariant compactly supported measure on G, and let $\psi(x)$ be $e^{-2\pi i x}$. Write

$$(\eta\cdot f)(x) = \int_G \eta(h) f(xh)\, \mathrm{d}h.$$

Using a standard pre–trace formula device, we have

$$\begin{aligned}
\int_{N\cap\Gamma\backslash N} (\eta\cdot f)(nx)\, \mathrm{d}n &= \int_{N\cap\Gamma\backslash N}\int_G \eta(h) f(nxh)\, \mathrm{d}h\, \mathrm{d}n = \int_{N\cap\Gamma\backslash N}\int_G \eta(x^{-1}n^{-1}h) f(h)\, \mathrm{d}h\, \mathrm{d}n \\
&= \int_{N\cap\Gamma\backslash N}\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \eta(x^{-1}n^{-1}\gamma h) f(h)\, \mathrm{d}h\, \mathrm{d}n \\
&= \int_{\Gamma\backslash G} f(h)\left(\int_{N\cap\Gamma\backslash N}\sum_{\gamma\in\Gamma}\eta(x^{-1}n^{-1}\gamma h)\, \mathrm{d}n\right) \mathrm{d}h.
\end{aligned}$$

Therefore,

$$\int_{N\cap\Gamma\backslash N} (\eta\cdot f)(nx)\cdot\psi(n)\, \mathrm{d}n = \int_{\Gamma\backslash G} f(h)\left(\int_{N\cap\Gamma\backslash N}\psi(n)\cdot\sum_{\gamma\in\Gamma}\eta(x^{-1}n^{-1}\gamma h)\, \mathrm{d}n\right) \mathrm{d}h.$$

2.1. L^2 Estimate on an Integral Kernel.

We denote the integral kernel

$$\int_{N\cap\Gamma\backslash N}\psi(n)\cdot\sum_{\gamma\in\Gamma}\eta(x^{-1}n^{-1}\gamma h)\, \mathrm{d}n$$

[4] Instead of obtaining such asymptotic solution of a second order ODE by Laplace transform and Watson's lemma, one can achieve a similar goal more easily by following G. D. Birkhoff and his pupils' approach, which is explained in the appendix.

by $q(h)$ and estimate the square of the L^2–norm of $q(h)$ as follows:

$$\begin{aligned} |q(h)|^2_{L^2(\Gamma\backslash G)} &= \int_{\Gamma\backslash G} |q(h)|^2 \, \mathrm{d}h \\ &\leq \int_{\Gamma\backslash G}\int_{N\cap\Gamma\backslash N}\int_{N\cap\Gamma\backslash N}\sum_{\gamma\in\Gamma}\sum_{\gamma_2\in\Gamma} |\psi(n)|\cdot|\psi(n_2)|\cdot\left|\eta(x^{-1}n^{-1}\gamma h)\right|\cdot\left|\eta(x^{-1}n_2^{-1}\gamma_2 h)\right| \, \mathrm{d}n\,\mathrm{d}n_2\,\mathrm{d}g \\ &\leq \int_G\int_C\int_C\sum_{\gamma\in\Gamma}\left|\eta(x^{-1}n^{-1}\gamma h)\right|\cdot\left|\eta(x^{-1}n_2^{-1}h)\right| \, \mathrm{d}n\,\mathrm{d}n_2\,\mathrm{d}g, \end{aligned}$$

where C is a large enough compact subset of N to surject to $N\cap\Gamma\backslash N$.

Let δ be the radius of the solid band $B_\delta(y)$ surrounding the orbit of iy under $N\cap\Gamma\backslash N$. Take η to be the characteristic function of $B_\delta(y)$, so that $\eta = |\eta|$. In order to obtain a further estimate on $|q(h)|^2_{L^2(\Gamma\backslash G)}$, we first claim that the sum over Γ appeared above is finite, and then give upper bounds for the two integrals where η's are involved.

Let Φ be the discrete subgroup of Γ defined and manipulated as follows:

$$\begin{aligned} \Phi &= \{\gamma\in\Gamma : \eta(x^{-1}n^{-1}\gamma h)\ \eta(x^{-1}n_2^{-1}h)\neq 0 \text{ for some } n, n_2\in C \text{ and } h\in\mathfrak{H}\} \\ &= \{\gamma\in\Gamma : x^{-1}n^{-1}\gamma h\in B_\delta(y),\ x^{-1}n_2^{-1}h\in B_\delta(y)\} \\ &= \{\gamma\in\Gamma : \gamma\in CxB_\delta(y)h^{-1},\ h\in CxB_\delta(y)\} \\ &= \{\gamma\in\Gamma : \gamma\in (CxB_\delta(y))\cdot(CxB_\delta(y))^{-1}\} \\ &\subset \Gamma\cap(CxB_\delta(y))\cdot(CxB_\delta(y))^{-1} \end{aligned}$$

which is an intersection of a discrete subgroup and a compact subset of G. Therefore, Φ is finite.

Now we deal with the integrals. Given $\gamma\in\Phi$ and $n\in C$, we integrate $\eta(x^{-1}n^{-1}\gamma h)$ with respect to h. Note that $\eta(x^{-1}n^{-1}\gamma h)\neq 0$ only for h in $B_\delta(y)$. Thus, this integral is dominated by δ^2. Given $h\in B_\delta(y)$, we integrate $\eta(x^{-1}n_2^{-1}h)$ with respect to n_2. Note that $\eta(x^{-1}n_2^{-1}h)\neq 0$ only for n_2^{-1} in $xB_\delta(y)B_\delta(y)^{-1}$. Thus, n_2^{-1} lies in $C\cap\big(xB_\delta(y)B_\delta(y)^{-1}\big)$. So this integral is dominated by δ. Therefore, we have

$$|q(g)|^2_{L^2(\Gamma\backslash G)} \ll_y \int_C \delta^3 \, \mathrm{d}n \ll_y \delta^3.$$

2.2. **Lower Bound on the Eigenvalues.** By the Plancherel theorem for L^2 Maass forms,

$$\sum_F |\langle q, F\rangle|^2 + \frac{|\langle q, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{2\pi}\int_0^\infty \left|\left\langle q, E_{\frac{1}{2}+it}\right\rangle\right|^2 \, \mathrm{d}t = |q|^2_{L^2(\Gamma\backslash G)} \ll_y \delta^3,$$

where F runs over an orthonormal basis of Maass cusp forms. Dropping the continuous spectrum part and the constant,

$$\sum_F |\langle q, F\rangle|^2 \ll_y \delta^3.$$

Therefore,

$$\sum_F \left|\int_{N\cap\Gamma\backslash N} (\eta\cdot F)(nx)\cdot\psi(n)\, \mathrm{d}n\right|^2 = \sum_F |\langle q, F\rangle|^2 \ll_y \delta^3.$$

It suffices to obtain a lower bound on the eigenvalues for η associated with F, using the unramified principal series model for the representation generated by F. In general, by the Casselman–Miličić subrepresentation theorem or the Harish–Chandra subquotient theorem (cf. [HC54] [CM82]), a spherical representation imbeds into (is also an image of) an unramified principal series. In the small rank–one cases, V. Bargmann used the asymptotics of solutions of second–order ODE to prove the subrepresentation theorem for these groups, which suffices for our purpose (cf. [Bar47]). Therefore, F generates principal series. By the K–invariance of η, $(\eta\cdot f)(x)$ is a right K–invariant vector, so that we can write $\eta\cdot F = \lambda_F(\eta)\cdot F$, where F is the unique (up to scalars) spherical vector in the irreducible spherical representation of G it generates. Let ϕ^o_s be the normalized spherical vector in the s^{th} unramified principal series:

$$I_s = \{\text{smooth } K\text{–finite } \phi :\ \phi\left(\begin{pmatrix} a & * \\ 0 & d\end{pmatrix} g\right) = \left|\frac{a}{d}\right|^{2s}\phi(g),\ s\in\mathbb{R}\}.$$

The following computation is carried out under the Cartan decomposition $G = KAK$. Since η is the characteristic function of $B_\delta(y)$ with radius $\delta \geq r$,

$$\eta\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{r/2} & \\ & e^{-r/2}\end{pmatrix}\right) = 1.$$

Observing

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{r/2} & \\ & e^{-r/2}\end{pmatrix} i = \begin{pmatrix}(\sin^2\theta e^r + \cos^2\theta e^{-r})^{-1/2} & * \\ 0 & (\sin^2\theta e^r + \cos^2\theta e^{-r})^{1/2}\end{pmatrix} i,$$

we have

$$\phi_s^o\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{r/2} & \\ & e^{-r/2}\end{pmatrix}\right) = \phi_s^o\begin{pmatrix}(\sin^2\theta e^r + \cos^2\theta e^{-r})^{-1/2} & * \\ 0 & (\sin^2\theta e^r + \cos^2\theta e^{-r})^{1/2}\end{pmatrix}.$$

Thus,

$$\begin{aligned}
\lambda_F(\eta) &= (\eta\cdot\phi_s^o)(1) = \int_G \eta(g)\cdot\phi_s^o(g)\ \mathrm{d}g\\
&= \int_{r\leq\delta}\eta\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{r/2} & \\ & e^{-r/2}\end{pmatrix}\right)\phi_s^o\left(\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}e^{r/2} & \\ & e^{-r/2}\end{pmatrix}\right)\ \mathrm{d}g\\
. &= \int_{r\leq\delta} 1\cdot\phi_s^o\begin{pmatrix}(\sin^2\theta e^r + \cos^2\theta e^{-r})^{-1/2} & * \\ 0 & (\sin^2\theta e^r + \cos^2\theta e^{-r})^{1/2}\end{pmatrix}\ \mathrm{d}g\\
&= \int_{r\leq\delta}\left|\frac{(\sin^2\theta e^r + \cos^2\theta e^{-r})^{-1/2}}{(\sin^2\theta e^r + \cos^2\theta e^{-r})^{1/2}}\right|^{2s}\cdot 1\ \mathrm{d}g = \int_{r\leq\delta}\frac{\mathrm{d}g}{(\sin^2\theta e^r + \cos^2\theta e^{-r})^{2s}}.
\end{aligned}$$

For $0 \leq r \leq \delta \ll 1/s$,

$$\begin{aligned}
\left(\sin^2\theta e^r + \cos^2\theta e^{-r}\right)^{2s} &\asymp \left(\sin^2\theta(1+r) + \cos^2\theta(1-r)\right)^{2s} = \left(1 + r\left(\sin^2\theta - \cos^2\theta\right)\right)^{2s}\\
&\leq (1+r)^{2s} \ll (1+r)^{1/\delta} \leq (1+r)^{1/r} < e \ll 1.
\end{aligned}$$

Therefore, we get a lower bound on $\lambda_F(\eta)$:

$$|\lambda_F(\eta)| = \left|\int_{B_\delta(y)}\phi_s^o(g)\ \mathrm{d}g\right| \gg \int_{B_\delta(y)} 1\ \mathrm{d}g \gg_y \delta^2 \qquad \text{for } |s_F| \ll 1/\delta.$$

Hence,

$$\sum_F\left|\int_{N\cap\Gamma\backslash N}\delta^2\cdot F(ng_y)\cdot\psi(n)\ \mathrm{d}n\right|^2 \ll_y \sum_F\left|\int_{N\cap\Gamma\backslash N}(\eta\cdot F)(nx)\cdot\psi(n)\ \mathrm{d}n\right|^2 \ll_y \delta^3 \qquad \text{for } |s_F| \ll 1/\delta,$$

where g_y sends the basepoint i to $iy \in \mathfrak{H}$. This yields

$$\sum_F|\theta(F)|^2 = \sum_F\left|\int_{N\cap\Gamma\backslash N}F(ng_y)\cdot\psi(n)\ \mathrm{d}n\right|^2 \ll_y \frac{\delta^3}{(\delta^2)^2} = \frac{1}{\delta} \qquad \text{for } |s_F| \ll 1/\delta.$$

Taking $T = 1/\delta$ gives the result. □

Remark 2.2.1. We are particularly interested in what happens when $|\lambda| \to \infty$, and this requires $\delta \to 0^+$, since $|\lambda| = |s||s-1| \leq |s|(|s|+1) \ll \delta^{-1}(\delta^{-1}+1)$. For fixed λ, the space of Maass forms is finite–dimensional, by Lax–Philips' argument on the compactness of the resolvent of the Friedrichs' extension of the Laplacian for example (cf. [LP77]).

Remark 2.2.2. If we didn't drop the continuous spectrum part and the constant appearing in the Plancherel for q, the proof would give a pre–trace formula estimate:

$$\sum_{F:|s_F|\ll T}|\theta(F)|^2 + \frac{|\theta(1)|^2}{\langle 1,1\rangle} + \frac{1}{2\pi}\int_0^T\left|\theta(E_{\frac12+it})\right|^2\ \mathrm{d}t \ll_y T,$$

which we will use later.

2.3. **Soft Estimate on the First Numerical Coefficient.** Given a Maass cusp form f, we give a soft estimate on its first numerical Fourier coefficients c_1. We use the soft estimate: $|\theta(f)|^2 \leq \sum_F |\theta(F)|^2$. Taking $T = s_f$ yields

$$|c_1 \cdot u(y, \lambda_f)| \ll_y \sqrt{|s_f|}.$$

In order to divide $|u|$, we have to use an asymptotic expansion of u in which the error term is smaller than the main term as $\lambda \to \infty$. Note that the usual asymptotic expansion

$$u(y, \lambda_f) \sim e^{-2\pi y}\left(1 + O\left(\frac{\lambda_f}{y}\right)\right)$$

which we mentioned earlier does not work. Now we write the Fourier expansion of f in terms of the modified Bessel function of the third kind of imaginary order:

$$f(x+iy) = \sum_{m \neq 0} c_n \sqrt{y} K_{i\nu}(2\pi|m|y) e^{2\pi m x}, \qquad \text{with } \nu = \sqrt{-\lambda_f - 1/4}.$$

Based on [Olv54], C. B. Balogh [Bal67] proved the following asymptotic result:

$$K_{i\nu}(2\pi y) = \frac{\pi\sqrt{2}}{\nu^{1/3}} e^{-\pi\nu/2}\left(\frac{\xi}{(2\pi y/\nu)^2 - 1}\right)^{1/4}\left(\mathrm{Ai}(\nu^{2/3}\xi) + O\left(\frac{1}{(1+\nu^{2/3}|\xi|)^{1/4}\nu}\right)\right),$$

uniform in $y > 0$ for $\nu > 1$, where $\mathrm{Ai}(y) \ll (1+|y|)^{-1/4}$ and $\frac{2}{3}\xi^{3/2} = ((2\pi y)^2 - 1)^{1/2} - \mathrm{arcsec}(2\pi y)$. We take an $M > 0$ such that

$$|K_{i\nu}(2\pi y)| \gg \left|\frac{1}{\nu^{1/3}} e^{-\pi\nu/2}\left(\frac{\xi}{(2\pi y/\nu)^2 - 1}\right)^{1/4} \cdot \left(\mathrm{Ai}(\nu^{2/3}\xi) + M \cdot \frac{1}{(1+\nu^{2/3}|\xi|)^{1/4}\nu}\right)\right|.$$

Dividing $|u|$ in the estimate on $|c_1 \cdot u(y, \lambda_f)|$ gives the result:

Corollary 2.3.1.

$$|c_1| \ll_y \frac{\sqrt{|s_F|}}{\sqrt{y}\left|\frac{1}{\nu^{1/3}} e^{-\pi\nu/2}\left(\frac{\xi}{(2\pi y/\nu)^2-1}\right)^{1/4} \cdot \left(\mathrm{Ai}(\nu^{2/3}\xi) + M \cdot \frac{1}{(1+\nu^{2/3}|\xi|)^{1/4}\nu}\right)\right|}.$$

Remark 2.3.1. As $\nu \to \infty$ ($\lambda \to -\infty$), this upper bound is not sharp. It is not entirely surprising that an estimate on a whole family of Maass cusp forms can't give a sharp estimate on individuals. However, we note that even such a result may not be easily obtained otherwise. Though this bound is weaker than that of J. Hoffstein and P. Lockhart, the proof is quite short and does not make use of many advanced tools.

Remark 2.3.2. In effect, the asmptotics of $K_{i\nu}(y)$ *in the transition range* suffices in a lot of situations in analytic number theory. Since it seems not easy to find this result in the literature, we state it here explicitly. Some care is necessary in this matter. Compare [IS95], [Iwa02] and [San04].

Suppose

$$\nu = y + ay^{1/3} + O(y^{-1/3}),$$

we have

$$K_{i\nu}(y) = \frac{2^{1/2}\pi}{3 \cdot y^{1/3}} e^{-\frac{1}{2}\pi\nu} f(a)\left(1 + O(y^{-2/3})\right),$$

where $f(a) = a^{1/2}\left(J_{1/3}\left(\frac{2^{3/2}a^{3/2}}{3}\right) + J_{-1/3}\left(\frac{2^{3/2}a^{3/2}}{3}\right)\right)$ with J the Bessel function of the first kind. This result was initially proved in the appendix of [Fri54].

3. Global Automorphic Levi–Sobolev Spaces

In this section, we set up the global automorphic Levi–Sobolev spaces. This is adapted from [Gar11b] with some rearrangements.

Remark 3.0.3. Notably, Colin de Verdière's meromorphic continuation of Eisenstein series can be understood through automorphic Levi–Sobolev spaces (cf. [CdV81] [Gar12b]). In addition, *Sobolev norms of automorphic functionals* serve as important tools in [BR99]. More generally, the idea of *distribution* has been successfully applied to the theory of automorphic forms. For instance, S. D. Miller and W. Schmid have done a series of important work by using *automorphic distributions.* Specifically, they obtained the first proof of the Voronoï summation formulas for $GL_3(\mathbb{Z})$ (cf. [MS04b] [MS04a] [MS06]), the complete Archimedean theory for the exterior square L–function for congruence subgroups of $GL_n(\mathbb{Z})$ (cf. [MS12] [MS08]) and so forth.

As on $\mathbb{R}^n$ (cf. [Bre11]), there are three useful characterizations of Levi–Sobolev spaces in our context. Let $U\mathfrak{g}^{\leq l}$ be the finite-dimensional subspace of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ whose degrees are less than or equal to l. Given $\alpha \in U\mathfrak{g}^{\leq l}$, $\nu_\alpha(f) = ||\alpha f||^2$ is a seminorm on $C_c^\infty(\Gamma\backslash G)$.

Definition 3.0.1. For $l \geq 0$,

$$H_c^l(\Gamma\backslash G) = \text{completion of } C_c^\infty(\Gamma\backslash G) \text{ with respect to } \nu_\alpha.$$

$$H^l(\Gamma\backslash G) = \text{completion of } C^\infty(\Gamma\backslash G) \text{ with respect to } \nu_\alpha.$$

$W^{2,l}(\Gamma\backslash G) = \{f \in L^2(\Gamma\backslash G) : \text{the distributional derivatives of f, } \alpha \circ f \text{ with } \alpha \text{ in } U\mathfrak{g}^{\leq l}\text{, lies in } L^2(\Gamma\backslash G)\}$.

3.1. **Spectral Parametrization and Spectral Transform.** We will prove shortly that these characterizations are actually identical. But perhaps more importantly, we will give an equivalent *spectral characterization* of global automorphic Levi–Sobolev spaces. This needs the notions of spectral parametrization and spectral transform introduced as follows.

Let

$$\Xi = \{\text{orthonormal basis of Maass cusp forms}\} \cup \{1\} \cup \left\{\frac{1}{2} + i[0,\infty)\right\}$$

and

$$\Phi_\xi = \begin{cases} F & \xi = F \\ 1/\langle 1,1\rangle^{1/2} & \xi = 1 \\ E_s & \xi = s \end{cases} \qquad \text{for } \xi \in \Xi.$$

We can rewrite the L^2 spectral decomposition

$$f = \sum_F \langle f, F\rangle F + \frac{\langle f,1\rangle \cdot 1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{-\infty}^{\infty} \left\langle f, E_{\frac{1}{2}+it}\right\rangle \cdot E_{\frac{1}{2}+it}\, \mathrm{d}t$$

as

$$f = \int_\Xi \langle f, \Phi_\xi\rangle \cdot \xi \, \mathrm{d}\xi.$$

The associated Plancherel is

$$|f|^2_{L^2(\Xi)} = \int_\Xi |\langle f, \Phi_\xi\rangle|^2 \, \mathrm{d}\xi.$$

The spectral transform

$$\mathscr{F} : C_c^\infty(\Gamma\backslash\mathfrak{H}) \to L^2(\Xi)$$

is initially given by integrating $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$ against F, 1 and $E_{\frac{1}{2}+it}$. By Plancherel,

$$\mathscr{F} : L^2(\Gamma\backslash\mathfrak{H}) \to L^2(\Xi)$$

is an isometric isomorphism, which is obtained by extension by continuity from $\mathscr{F}$ on $C_c^\infty(\Gamma\backslash\mathfrak{H})$. The *spectral characterization* of global automorphic Levi–Sobolev spaces, which we shall obtain later, is

$$H^l(\Gamma\backslash\mathfrak{H}) = \{f \in L^2(\Gamma\backslash\mathfrak{H}) : (1-\lambda)^{l/2}\mathscr{F}f \in L^2(\Xi)\}.$$

3.2. **Equivalence of Three Classical Characterizations.** We first show that the three characterizations of automorphic Levi–Sobolev spaces are identical.

Theorem 3.2.1. $H^l_c(\Gamma\backslash\mathfrak{H}) = H^l(\Gamma\backslash\mathfrak{H}) = W^{2,l}(\Gamma\backslash\mathfrak{H})$.

Proof. First we show $H^l_c(\Gamma\backslash\mathfrak{H}) = H^l(\Gamma\backslash\mathfrak{H})$. It suffices to show $H^l_c(\Gamma\backslash G) = H^l(\Gamma\backslash G)$, and take right K–invariant elements in the end. Let $G = NAK$ be the Iwasawa decomposition. Let η be a family of smooth cut–off functions which are both N– and K–invariant. Also, we require $\alpha\eta$ be bounded independent of the support of η.[5] We claim that the pointwise multiplication $\eta \cdot f$ approaches any given f in $H^l(\Gamma\backslash G)$. By Leibniz' rule on differentiation with respect to product of two functions, the derivative of $\eta \cdot f$ is a finite linear combination of elements of the form $\alpha\eta \cdot \beta f$ with $\alpha \in U\mathfrak{g}^{\leq l_1}$ and $\beta \in U\mathfrak{g}^{\leq l_2}$ such that $l_1 + l_2 = l$. So the L^2 norm of $\alpha(\eta - 1) \cdot \beta f$ is dominated by the L^2 norm of βf over $(\Gamma\backslash G - \mathrm{supp}\eta)$. As the support of η approaches $\Gamma\backslash G$, this integral will vanish.

Next, we prove $H^l_c(\Gamma\backslash\mathfrak{H}) = W^{2,l}(\Gamma\backslash\mathfrak{H})$. Uryshon's lemma and a reasonable definition of integral ensure that $C^0_c(\Gamma\backslash G)$ is dense in $L^2(\Gamma\backslash G)$.[6] Therefore, G acts on $L^2(\Gamma\backslash G)$ *continuously* by right translation. In fact, the continuity of the action of G on $L^2(\Gamma\backslash G)$ implies the continuity of the action of G on $W^{2,l}(\Gamma\backslash G)$, since G maps bounded subsets of $U\mathfrak{g}^{\leq l}$ to bounded subsets, hence preserving the topology given by seminorms $\nu_\alpha(f) = \sup_{\beta\in B} \nu_B(f)$, where B runs over bounded subsets of $U\mathfrak{g}^{\leq l}$.

From the continuity of the right action of G, we claim that test functions are dense in $W^{2,l}(\Gamma\backslash G)$. Let $\{\eta\}$ be a smooth approximate identity. Given η in $W^{2,l}(\Gamma\backslash G)$, the following lemma[7] implies that

$$\eta \cdot u = \int_G \eta(g) g \cdot u \, \mathrm{d}g \longrightarrow u \qquad \text{in the topology of } W^{2,l}(\Gamma\backslash G),$$

where $(g \cdot u)(x) = u(xg)$.

Lemma 3.2.2. *Let G be a separable, locally compact, Hausdorff topological group acting continuously on V, a locally convex, quasi-complete topological vector space. For $\{\psi_i\}$ an approximate identity and $F : G \to V$ a continuous vector-valued function, we have*

$$\int_G \psi_i(g) F(g) \, \mathrm{d}g \longrightarrow F(1) \qquad \textit{in the topology of } V.$$

Proof. Note that $\psi_i(g)\,\mathrm{d}g$ gives a probablity measure on the support of ψ_i, since the integral of ψ_i is 1. F is a continuous V–valued function on this measure space. By the basic estimate[8] on *Gelfand–Pettis integral* (cf. [Gel36] [Pet38]), the integral lies in the closure of the convex hull of F(support of ψ_i). The following argument is an analogue of the $\varepsilon/2$–argument in metric space, rewritten in the language of topological vector spaces. Given a convex neighbourhood N of 0 in V, by the continuity of F, there exists a sufficiently small neighbourhood U of 1 in G such that for every $h \in U$, $F(h)$ lies in $F(1) + N/2$. The local compactness of G implies that there exists a smaller neighbourhood U' of 1 in G such that its closure $\overline{U'}$ is compact and contained in U. Thus, by continuity of F, $F(\overline{U'})$ is compact and contained in $F(1) + N/2$. The local convexity of $N/2$ implies that the convex hull of $F(\overline{U'})$ in contained in $F(1) + N/2$. Hence, the closure of the convex hull of $F(\overline{U'})$ is contained in $F(1) + N/2 + N/2 = F(1) + N$.

□

It now suffices to show that $\eta \cdot u$ is a smooth vector and hence[9] a smooth function. Given $v \in \mathfrak{g}$, $f \in C^0_c(\Gamma\backslash G)$ and $t \in \mathbb{R}$, we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} e^{tv} \cdot (\eta \cdot f) = \frac{\partial}{\partial t}\bigg|_{t=0} e^{tv} \cdot \int_G \eta(g) g \cdot f \, \mathrm{d}g = \frac{\partial}{\partial t}\bigg|_{t=0} \int_G \eta(g) \left(e^{tv} \cdot g\right) \cdot f \, \mathrm{d}g.$$

---

[5]For instance, as suggested in [Gar11b], one may take η to be a function which vanishes for $\log y < R$, is smooth for $R \leq \log y \leq R + 1$, and is 1 for $\log y > R + 1$. In this example, the derivative of η should come from the Lie algebra of A. So $y \frac{\partial}{\partial y} \eta(\log y - R) = \eta'(\log y - R)$, by design.

[6]A proof of the general topological space case appears in the appendix of [Gar13].

[7]This lemma is also very useful in a proof of the uniqueness of Haar measure given by P. Garrett; see [Gar05].

[8]Let X be a locally compact, Hausdorff topological space with a finite, positive, Borel measure. Let V be a locally convex, quasi-complete topological vector space. The Gelfand–Pettis integral $\int_X f$ lies in the closure of the convex hull of $f(X)$ with a multiple of the measure of X. See [Gar12c] for more detail.

[9]Note that this may not be true when the representation space is a general function space.

By the smoothness of $e^{tv} \cdot g$ and Urysohn's lemma, $H^l_c(\Gamma\backslash G) = W^{2,l}(\Gamma\backslash G)$.

□

3.3. **Termwise L^2–differentiation.** We need a correct notion of differentiation here, that is, *termwise L^2–differentiation.* L^2 differentiation on $\Gamma\backslash\mathfrak{H}$ by Δ (for non–negative index Levi–Sobolev spaces) is obtained as follows. $\Delta : H^l(\Gamma\backslash G) \cap C_c^\infty(\Gamma\backslash G) \to H^{l-2}(\Gamma\backslash G) \cap C_c^\infty(\Gamma\backslash G)$ extends by continuity to $H^l(\Gamma\backslash G) \to H^{l-2}(\Gamma\backslash G)$, for $l \geq 2$. (Δ can descend to $H^l(\Gamma\backslash\mathfrak{H}) \to H^{l-2}(\Gamma\backslash\mathfrak{H})$, since Δ commutes with K.) We illustrate *termwise* L^2–differentiation by *spectral expansions*. We mentioned earlier that by Plancherel, $\mathscr{F} : L^2(\Gamma\backslash\mathfrak{H}) \to L^2(\Xi)$ is an *isometric* isomorphism. Therefore, given $f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$,

$$\mathscr{F}\Delta f = \int_{\Gamma\backslash\mathfrak{H}} \Delta f \Phi_\xi = \int_{\Gamma\backslash\mathfrak{H}} f \Delta\Phi_\xi = \int_{\Gamma\backslash\mathfrak{H}} f\left(\lambda_\xi \Phi_\xi\right) = \lambda_\xi \int_{\Gamma\backslash\mathfrak{H}} f\Phi_\xi = \lambda_\xi \mathscr{F} f.$$

Then by extension by continuity, we have

$$\Delta f = \mathscr{F}^{-1}\mathscr{F}\Delta f = \mathscr{F}^{-1}\lambda\mathscr{F} f, \qquad \text{for } f \in H^l(\Gamma\backslash\mathfrak{H}) \text{ with } l \geq 2.$$

Thus, *termwise* L^2–differentiation (for non–negative index Levi–Sobolev spaces) is valid.

3.4. **Spectral Characterization.** Now we characterize global automorphic Levi–Sobolev spaces by spectral expansions. We first observe that the topology of $H^l(\Gamma\backslash G)$ can also be given equivalently by Casimir (or Laplacian).

Theorem 3.4.1. *$-\Omega + 1$ and $\alpha \in U\mathfrak{g}^{\leq n}$ induce the same topology on $H^l(\Gamma\backslash\mathfrak{H})$.*

Proof. Given *spherical* f in $C_c^\infty(\Gamma\backslash G)$, we claim

$$\langle \alpha f, \alpha f\rangle \ll \langle (-\Omega + 1)^n f, f\rangle .$$

Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition. Given $\alpha \in U\mathfrak{g}$, we can write $\alpha = x_1 \cdots x_n y_1 \cdots y_m$ with $x_i \in \mathfrak{p}$ and $y_j \in \mathfrak{k}$, by the Poincaré–Birkhoff–Witt theorem. Since f is spherical, we may just suppose $\alpha = x_1 \cdots x_n$. Note that $-\Omega_\mathfrak{p}$ is a non–negative symmetric operator, since the Killing form is positive–definite on $\mathfrak{p}$. Similarly, $-\Omega_\mathfrak{k}$ is a non–positive symmetric operator. For $n = 1$, we extend x_1 to be a self–dual basis $\{X_i\}$ of $\mathfrak{p}$ such that $x_1 = X_1$.

$$\langle x_1 f, x_1 f\rangle = \int x_1 f \cdot \overline{x_1 f} \leq \sum_i X_i f \cdot \overline{X_i f} = \int \left(\sum_i -X_i^2\right) f \cdot \overline{f} = \int -\Omega_\mathfrak{p} f \cdot \overline{f} = \langle -\Omega_\mathfrak{p} f, f\rangle .$$

Given $n = \deg\alpha$, we can find a sufficiently[10] large positive constant such that $T = -\Omega_\mathfrak{p} - \Omega_\mathfrak{k} + C$ is a strictly positive (unbounded) operator on the Hilbert–space closure of

$$U\mathfrak{g}^{\leq n} \cdot \{f \in C_c^\infty(\Gamma\backslash G) : f \text{ is spherical}\}.$$

By the construction of the Friedrichs' self–adjoint extension (cf. [Gar11d] [Lax02]), there exists a positive, symmetric and bounded inverse R of T. By the spectral theory of bounded symmetric operators on Hilbert spaces, there is a positive symmetric $\sqrt{R}$ in the closure of the polynomial algebra $\mathbb{C}[R]$. Therefore, $\sqrt{T} = 1 - \sqrt{R}$ is a positive symmetric operator which commutes with all elements in $U\mathfrak{g}^{\leq n}$. By induction on n,

$$\begin{aligned} \langle (-\Omega + C)\alpha f, \alpha f\rangle &= \left\langle \sqrt{T}\sqrt{T}\alpha f, \alpha f\right\rangle = \left\langle \sqrt{T}\alpha f, \sqrt{T}\alpha f\right\rangle = \left\langle \alpha\sqrt{T} f, \alpha\sqrt{T} f\right\rangle \\ &\leq \left\langle T^n\sqrt{T} f, \sqrt{T} f\right\rangle = \left\langle T^{n+1} f, f\right\rangle . \end{aligned}$$

Hence,

$$\langle \alpha f, \alpha f\rangle \leq \langle T^n f, f\rangle \ll \langle (T+1)^n f, f\rangle .$$

□

[10]Indeed, it suffices to let C be strictly larger than the largest eigenvalue of $\Omega_\mathfrak{k}$ over $U\mathfrak{g}^{\leq n}$.

Definition 3.4.1. For $l \geq 0$, define

$$V^l = \{\text{measurable v} : (1-\lambda)^{l/2} v \in L^2(\Xi)\},$$

with the Hilbert–space structure from the norm

$$|v|_{V^l} = \left(\int_\Xi (1-\lambda)^l |v|^2 \right)^{1/2}.$$

Theorem 3.4.2. *$\mathscr{F} : H^{2l}(\Gamma\backslash\mathfrak{H}) \to V^{2l}$ is an isometric isomorphism, for $l \geq 0$.*

Proof. By Plancherel, $\mathscr{F}$ is an L^2–isometry. Given $f \in H^{2l}(\Gamma\backslash\mathfrak{H})$, by the notion of termwise L^2–differentiation and the Plancherel theorem,

$$|(1-\Delta)^l f|_{L^2(\Gamma\backslash\mathfrak{H})} = |\mathscr{F}(1-\Delta)^l f|_{L^2(\Xi)} = |(1-\lambda)^l \mathscr{F} f|_{L^2(\Xi)}.$$

That is, $\mathscr{F} f \in V^{2l}$. Conversely, given $v \in V^{2l}$ and a test function ϕ,

$$\begin{aligned} \left((1-\Delta)\mathscr{F}^{-1} v\right)(\phi) &= \mathscr{F}^{-1} v\left((1-\Delta)(\phi)\right) = v\left(\mathscr{F}((1-\Delta)\phi)\right) \\ &= v\left((1-\lambda)\mathscr{F}\phi\right) = (1-\lambda) v\left(\mathscr{F}\phi\right) \\ &= \left(\mathscr{F}^{-1}\left((1-\lambda)v\right)\right)(\phi), \end{aligned}$$

where all equalities are justified by Plancherel except the third one which is by the fact that Fourier transform interchanges differentiation with multiplication on test functions. That is, the inverse Fourier transform interchanges *distributional* differentiation with multiplication on V^l,

$$(1-\Delta)\mathscr{F}^{-1} v = \mathscr{F}^{-1}\left((1-\lambda)v\right).$$

Therefore, given $0 \leq k \leq l$,

$$(1-\Delta)^k \mathscr{F}^{-1} v = \mathscr{F}^{-1}(1-\lambda)^k v \in \mathscr{F}^{-1} V^{2l-2k} \subset \mathscr{F}^{-1} L^2 = L^2.$$

Since all such distributional derivatives lie in L^2, $\mathscr{F}^{-1} v$ lies in $H^{2l}(\Gamma\backslash\mathfrak{H})$. $\square$

As a corollary, we obtain the spectral characterization of global automorphic Levi–Sobolev spaces.

Corollary 3.4.3. $H^l(\Gamma\backslash\mathfrak{H}) = \{f \in L^2(\Gamma\backslash\mathfrak{H}) : (1-\lambda)^{l/2}\mathscr{F} f \in L^2(\Xi)\}$.

3.5. **Negative–index Levi–Sobolev Spaces.** As expected, we characterize negative–index global Levi–Sobolev spaces as Hilbert–space duals of positive ones. The continuous L^2–differentiation $\Delta : H^l(\Gamma\backslash\mathfrak{H}) \to H^{l-2}(\Gamma\backslash\mathfrak{H})$ (for $l \geq 2$) gives an adjoint[11] on negative–index spaces (still denoted by Δ):

$$\Delta : H^{-l+2}(\Gamma\backslash\mathfrak{H}) \longrightarrow H^{-l}(\Gamma\backslash\mathfrak{H}) \qquad \text{for } l \geq 2.$$

Similarly, we characterize V^{-l} as the Hilbert–space dual of V^l, with the complex–bilinear pairing $\langle\ ,\ \rangle_{V^l \times V^{-l}}$. By the self-duality of V^0, we have the following chain of inclusions

$$\cdots \subset V^2 \subset V^1 \subset V^0 \subset V^{-1} \subset V^{-2} \subset \cdots,$$

where V^l is dense in V^{l-1} for all $l \in \mathbb{Z}$. In addition, let $\mu : V^l \to V^{l-2}$ be the multiplication operator which multiplies functions in V^l by $1-\lambda$. Indeed, μ is a Hilbert–space isomorphism, by comparing the corresponding norms with those in L^2.

Theorem 3.5.1. *The following diagram commutes.*

$$\begin{tikzcd} \cdots \arrow[r, "1-\Delta"] & H^4 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^2 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^0 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^{-2} \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^{-4} \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & \cdots \\ \cdots \arrow[r, "\times(1-\lambda)"] & V^4 \arrow[r, "\times(1-\lambda)"] & V^2 \arrow[r, "\times(1-\lambda)"] & V^0 \arrow[r, "\times(1-\lambda)"] & V^{-2} \arrow[r, "\times(1-\lambda)"] & V^{-4} \arrow[r, "\times(1-\lambda)"] & \cdots \end{tikzcd}$$

Therefore,

$$\mathscr{F}\left((1-\Delta)f\right) = (1-\lambda)\mathscr{F} f,$$

for f in any global automorphic Levi–Sobolev space.

[11] Specifically, given a test function ϕ, $\Delta^* \langle f, \phi\rangle = \langle f, \Delta\phi\rangle$, for $f \in H^l(\Gamma\backslash\mathfrak{H})$ with $l \geq 2$. Indeed, this is the usual definition of differentiation on distributions.

Proof. By Plancherel, the complex–bilinear adjoint of $\mathscr{F} : L^2(X) \to L^2(\Xi)$ is $\mathscr{F}^{-1}$. Therefore, we can glue the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "1-\Delta"] & H^4 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^2 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^0 \arrow[d, "\mathscr{F}"', "\approx"] \\
\cdots \arrow[r, "\times(1-\lambda)"] & V^4 \arrow[r, "\times(1-\lambda)"] & V^2 \arrow[r, "\times(1-\lambda)"] & V^0
\end{tikzcd}$$

to the adjoint diagram

$$\begin{tikzcd}
H^0 \arrow[r, "1-\Delta"] & H^{-2} \arrow[r, "1-\Delta"] & H^{-4} \arrow[r, "1-\Delta"] & \cdots \\
V^0 \arrow[u, "\mathscr{F}^{-1}", "\approx"'] \arrow[r, "\times(1-\lambda)"] & V^{-2} \arrow[u, "\mathscr{F}^{*}", "\approx"'] \arrow[r, "\times(1-\lambda)"] & V^{-4} \arrow[u, "\mathscr{F}^{*}", "\approx"'] \arrow[r, "\times(1-\lambda)"] & \cdots
\end{tikzcd}$$

yielding

$$\begin{tikzcd}
\cdots \arrow[r, "1-\Delta"] & H^4 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^2 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^0 \arrow[r, "1-\Delta"] \arrow[d, leftrightarrow] & H^{-2} \arrow[r, "1-\Delta"] & H^{-4} \arrow[r, "1-\Delta"] & \cdots \\
\cdots \arrow[r, "\times(1-\lambda)"] & V^4 \arrow[r, "\times(1-\lambda)"] & V^2 \arrow[r, "\times(1-\lambda)"] & V^0 \arrow[r, "\times(1-\lambda)"] & V^{-2} \arrow[u, "\mathscr{F}^{*}", "\approx"'] \arrow[r, "\times(1-\lambda)"] & V^{-4} \arrow[u, "\mathscr{F}^{*}", "\approx"'] \arrow[r, "\times(1-\lambda)"] & \cdots
\end{tikzcd}$$

Note that $(\mathscr{F}^*)^{-1}$ is given by extension by continuity from $\mathscr{F}$ on test functions, namely, given a test function ϕ, $\mathscr{F}^* \langle f, \mathscr{F}\phi \rangle = \langle f, \phi \rangle$ for every f in $H^{2l}(\Gamma\backslash\mathfrak{H})$ for $l \geq 0$. Thus, we *define* $(\mathscr{F}^*)^{-1}$ as $\mathscr{F}$, so that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r, "1-\Delta"] & H^4 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^2 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^0 \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^{-2} \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & H^{-4} \arrow[r, "1-\Delta"] \arrow[d, "\mathscr{F}"', "\approx"] & \cdots \\
\cdots \arrow[r, "\times(1-\lambda)"] & V^4 \arrow[r, "\times(1-\lambda)"] & V^2 \arrow[r, "\times(1-\lambda)"] & V^0 \arrow[r, "\times(1-\lambda)"] & V^{-2} \arrow[r, "\times(1-\lambda)"] & V^{-4} \arrow[r, "\times(1-\lambda)"] & \cdots
\end{tikzcd}$$

□

4. Pre–trace Formula and Global Automorphic Levi–Sobolev Norm

In this section, we show that the distribution θ, evaluation map of the first Fourier coefficient of Maass cusp form at height $y > 1$, lies in $H^{-1/2-\varepsilon}(\Gamma\backslash\mathfrak{H})$ for all $\varepsilon > 0$, by using the connection between the pre–trace formula estimate we obtained earlier and the global automorphic Levi–Sobolev norm. In particular, this legitimizes computation of θ on any automorphic forms that are in $+l$ global automorphic Levi–Sobolev space for $l > 1/2$, so that the interchanging of θ with integrals is not a problem for instance.

We first need two lemmas proved by a standard device in analytic number theory.

Lemma 4.0.2. *Suppose* $\int_1^T |f(t)|^2 \, \mathrm{d}t \ll T^r$. *We have*

$$\int_1^T \frac{|f(t)|^2}{t^{2h}} \, \mathrm{d}t \ll T^{r-2h+\varepsilon} \qquad \textit{for every } \varepsilon > 0.$$

In addition, for $h > \frac{r}{2}$, *the integral is finite.*

Proof.

$$\begin{aligned}\int_1^T \frac{|f(t)|^2}{t^{2h}}\ \mathrm{d}t &= \sum_{n=0}^{N-1}\int_{T^{\frac{n}{N}}}^{T^{\frac{n+1}{N}}} \frac{|f(t)|^2}{t^{2h}}\ \mathrm{d}t \le \sum_{n=0}^{N-1}\frac{\int_1^{T^{\frac{n+1}{N}}} |f(t)|^2\ \mathrm{d}t}{\left(T^{\frac{n}{N}}\right)^{2h}} \ll \sum_{n=0}^{N-1}\frac{\left(T^{\frac{n+1}{N}}\right)^r}{\left(T^{\frac{n}{N}}\right)^{2h}}\\ &= T^{\frac{r}{N}}\cdot \sum_{n=0}^{N-1} T^{n\cdot\frac{r-2h}{N}} = T^{\frac{r}{N}}\cdot\frac{T^{N\cdot\frac{r-2h}{N}}-1}{T^{\frac{r-2h}{N}}-1}\\ &= \frac{T^{r-2h+\frac{2h}{N}}-T^{\frac{2h}{N}}}{1-T^{-\frac{r-2h}{N}}} \ll_{r,h,N} T^{r-2h+\varepsilon},\end{aligned}$$

where we take N sufficiently large such that $2h/N < \varepsilon$ for every $\varepsilon > 0$. □

Lemma 4.0.3. $\sum_{|s_F|\ll T} |F|^2 \ll T^r$ *implies*

$$\sum_{|s_F|\ll T} \frac{|F|^2}{(1+s_F^2)^{2h}} \ll T^{r-2h+\varepsilon} \qquad \text{for every } \varepsilon > 0.$$

Proof. Replace integrals by sums in the proof above. □

Theorem 4.0.4. *The distribution θ that evaluates the first Fourier coefficient of Maass cusp form at height $y > 1$ lies in $H^{-1/2-\varepsilon}(\Gamma\backslash\mathfrak{H})$ for every $\varepsilon > 0$.*

Proof. An automorphic spectral decomposition of θ is of the form

$$\theta = \sum_F \theta(F)\cdot F + \frac{\theta(1)}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{-\infty}^{\infty} \theta(E_{\frac12+it})\cdot E_{\frac12+it}\ \mathrm{d}t.$$

We show that this expansion converges in $H^{-1/2-\varepsilon}$ toplogy. It certainly does not converge pointwise, nor in L^2.

Recall the pre–trace formula estimate we obtained earlier:

$$\sum_{F:|s_F|\ll T} |\theta(F)|^2 + \frac{|\theta(1)|^2}{\langle 1,1\rangle} + \frac{1}{2\pi}\int_0^T \left|\theta(E_{\frac12+it})\right|^2\ \mathrm{d}t \ll_y T.$$

By the lemmas above, the square of the $-1/2-\varepsilon$ global automorphic Levi–Sobolev norm of θ is finite:

$$\begin{aligned}|\theta|^2_{H^{-\frac12-\varepsilon}(\Gamma\backslash\mathfrak{H})} &= \sum_F \frac{|\theta(F)|^2}{(1+|s_F|^2)^{1/2+\varepsilon}} + \frac{|\theta(1)|^2}{\langle 1,1\rangle} + \frac{1}{2\pi i}\int_0^\infty \frac{\left|\theta(E_{\frac12+it})\right|^2}{(1+t^2)^{1/2+\varepsilon}}\ \mathrm{d}t\\ &= \lim_{T\to\infty}\left(\sum_{F:|s_F|\le T} \frac{|\theta(F)|^2}{(1+|s_F|^2)^{1/2+\varepsilon}} + \frac{|\theta(1)|^2}{\langle 1,1\rangle} + \frac{1}{2\pi i}\int_0^T \frac{\left|\theta(E_{\frac12+it})\right|^2}{(1+t^2)^{1/2+\varepsilon}}\ \mathrm{d}t\right)\\ &\ll_y \lim_{T\to\infty} T^{-\varepsilon} < \infty.\end{aligned}$$

In this case, $r = 1$ so that we take $h = 1/2+\varepsilon$ yielding $r - 2h + \varepsilon = -\varepsilon$. □

Remark 4.0.1. A similar argument can show that the automorphic Dirac delta lies in $H^{-1-\varepsilon}(\Gamma\backslash\mathfrak{H})$, by the standard pre–trace formula estimate (cf. [Iwa02]):

$$\sum_{F:|s_F|\ll T} |F(z_0)|^2 + \frac{1}{4\pi}\int_{\frac12-iT}^{\frac12+iT} |E_s(z_0)|^2\ \mathrm{d}s \ll_C T^2,$$

for z_o in a fixed compact subset C of $\mathfrak{H}$.

5. The Utility of Levi–Sobolev Spaces and Other Modern Analysis in the Spectral Theory of Automorphic Forms

P. Garrett has emphasized the usefulness of L^2 Levi–Sobolev spaces and Schwartz' distributions in clarifying convergence of automorphic spectral expansions. Further, rather than constructing solutions to differential equations by solving the equation on G/K and forming Poincaré series, whose convergence is its own problem, and analyzing analytic continuation of the Poincaré series via their Fourier expansions, spectral expansions can be used directly. This idea is not without precedent.

In 1977, H. Haas [Has77] attempted to numerically compute the eigenvalues of Δ on $\Gamma\backslash\mathfrak{H}$. H. Stark observed some zeros of the Riemann zeta function in Haas' list, and D. Hejhal noticed several zeros of $L(s,\chi)$ in it where χ is the non-trivial Dirichelet character mod 3. As Hejhal [Hej81] observed, the correct interpretation of Haas' numerical result is that, for the eigenvalue problem

$$(\Delta - \lambda_w)u_w = \delta_\omega^{\mathrm{afm}},$$

zeros of ζ and L–functions appear in the list of w's, with ω a cube root of unity and $\delta_\omega^{\mathrm{afm}}$ the automorphic Dirac delta at ω.

Towards the Riemann Hypothesis through the Hilbert–Pólya approach, we want the operator to be self–adjoint so that the eigenvalue $\lambda_w = w(w-1)$ would be real and non–positive, and as a result we would have $\mathrm{Re}(w) = \frac{1}{2}$ or $w \in [0,1]$. A direct but insufficient approach is to directly take the Friedrichs' extension of Δ in the eigenvalue problem above. As noted in the previous section, $\delta_\omega^{\mathrm{afm}}$ lies in $H^{-1-\varepsilon}(\Gamma\backslash\mathfrak{H})$ which is not a continuous functional on $H^1(\Gamma\backslash\mathfrak{H})$. Nevertheless, the construction of the Friedrich's extension requires the eigenfunction of the extended operater be in $H^1(\Gamma\backslash\mathfrak{H})$.

Y. Colin de Verdière [CdV83] speculates about the eigenvalue problem $(\Delta - \lambda_w)u_w = \delta_{z_0}^{\mathrm{nc}}$ in $L^2_{\mathrm{nc}}(\Gamma\backslash\mathfrak{H}) = L^2_{\mathrm{cusp}}(\Gamma\backslash\mathfrak{H})^\perp$, where $\delta_{z_0}^{\mathrm{nc}}(f)$ is the orthogonal projection of $\delta_\omega^{\mathrm{afm}}$ to the space spanned by pseudo–Eisenstein series and constants. E. Bombieri and Garrett observe that $\delta_{z_0}^{\mathrm{nc}}$ is in $H^{-3/4+\varepsilon}(\Gamma\backslash\mathfrak{H})$ (which is contained in $H^{-1}(\Gamma\backslash\mathfrak{H})$) because of the Hardy–Littlewood second–moment asymptotic result for $\zeta(s)$ (cf. [HL16]). So if we take the Friedrichs' extension (denoted by $\widetilde{\Delta}$) of Δ in this eigenvalue problem, u_w would be in $H^1(\Gamma\backslash\mathfrak{H})$. This, together with the boundary condition $\delta_{z_0}^{\mathrm{nc}}(u_w) = 0$, ensure that u_w is a genuine eigenfuction for the self–adjoint operator $\widetilde{\Delta}$.

More generally and specifically, for a fundamental discriminant $d < 0$, we let H_d be a set of Heegner points in $\mathfrak{H}$ representing the ideal classes of the ring of integers in $\mathbb{Q}(\sqrt{d})$. Define

$$\theta_d = \sum_{z_o \in H_d} \delta_{z_0}^{nc}.$$

For $d < -4$ (to avoid roots of unity), we have

$$\theta_d E_w = \left(\frac{\sqrt{d}}{2}\right)^w \frac{\zeta_{\mathbb{Q}(\sqrt{d})}(w)}{\zeta(2w)} = \left(\frac{\sqrt{d}}{2}\right)^w \frac{\zeta(w)L(w,\chi_d)}{\zeta(2w)},$$

by the quadratic reciprocity law, where χ_d is the quadratic character attached to $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. For simplicity, we denote θ_d by θ. Let Δ_θ denote the restriction of Δ to $L^2_{\mathrm{nc}}(\Gamma\backslash\mathfrak{H}) \cap C_c^\infty(\Gamma\backslash\mathfrak{H})$ with the boundary condition $\theta(u_w) = 0$. Therefore, the domain of Δ_θ is $L^2_{\mathrm{nc}}(\Gamma\backslash\mathfrak{H}) \cap C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap \ker\theta$.

Remark 5.0.2. $\theta \in H^{-1}(\Gamma\backslash\mathfrak{H})$ ensures that $\widetilde{\Delta}_\theta$ does not forget the boundary condition entirely, though does overlook it. That is, if we apply an automorphic Dirac delta to the whole spectrum, it is ignored entirely on the cuspidal spectrum. But the Friedrichs' extension of the restriction of the Laplacian on the non–cuspidal part works well.

By solving the differential equation $(\Delta - \lambda_w)u_w = \theta$ through spectral expansions instead of Fourier expansions, Bombieri and Garrett proved the following inclusion in a clean way:

$$\{w : \lambda_w \text{ is an eigenvalue of } \widetilde{\Delta}_\theta\} \subset \{w : \mathrm{Re}(w) = \frac{1}{2} \text{ or } w \in [0,1] \text{ such that } \zeta_{\mathbb{Q}(\sqrt{d})}(w) = 0\}.$$

Remark 5.0.3. As an example, we show how to solve $(\Delta - \lambda_w)u_w = \theta$ by spectral expansions. The automorphic spectral expansion of θ is

$$\theta = \frac{\theta(1)\cdot 1}{\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \theta(E_{1-s})\cdot E_s \,\mathrm{d}s, \quad \text{in } H^{-1}\text{–topology}.$$

By division,

$$u_w = \frac{\theta(1)\cdot 1}{(\lambda_1-\lambda_w)\,\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \frac{\theta(E_{1-s})\cdot E_s}{\lambda_s-\lambda_w}\,\mathrm{d}s, \quad \text{in } H^1\text{–topology for } \mathrm{Re}(w) > \frac{1}{2}.$$

We note that the notion of termwise L^2–differentiation mentioned earlier is crucial here. Unlike distributional differentiation in the usual sense, it tracks Sobolev index.

Appendix A. Asymptotic Expansion of Integer–translation Invariant Eigenfunction

An integer–translation invariant eigenfunction f of $\Delta = y^2\left(\partial_{xx}+\partial_{yy}\right)$ on $\mathfrak{H}$ is of the form $f(x+iy) = u(y,\lambda)e^{2\pi i x}$. For simplicity, we write $u(y,\lambda)$ as $u(y)$. $(\Delta-\lambda)u(y)e^{2\pi i x} = 0$ gives

$$y^2 u''(y) - (4\pi^2 y^2 + \lambda)u(y) = 0.$$

We first discuss the asymptotic behaviour of $u(y)$ as $y \to 0^+$, so that we can introduce some relevant concepts along the way. Indeed, $y=0$ is not an *ordinary point*[12], but a *regular singular point*[13].

Remark A.0.4. In fact, the solution of a general Euler–Cauchy type ODE

$$y^2u''(y) + y\cdot b(y)u'(y) + c(y)u(y) = 0$$

with a regular singular point at 0 is aymptotic to the solution of

$$y^2u''(y) + y\cdot b(0)u'(y) + c(0)u(y) = 0$$

as $y \to 0^+$, as is correctly suggested by the heuristic of *freezing* $b(y)$ and $c(y)$ at 0^+ (cf. [Erd10] [Gar11e]).

Therefore, we can *freeze* our equation at $y=0$, yielding

$$y^2u''(y) - \lambda u(y) = 0.$$

We parameterize λ to be $s(s-1)$ so that the solution space of the frozen equation is spanned by y^s and y^{1-s} for $s \neq 1/2$. Hence, as $y \to 0^+$,

$$u(y) = Ay^s\left(1+O(y)\right) + By^{1-s}\left(1+O(y)\right),$$

for some $A, B \in \mathbb{R}$.

Now we establish the asymptotic expansion of $u(y)$ as $y \to \infty$. Note that $y=\infty$ is neither a regular singular point nor an ordinary point. To see this, let $z = 1/y$ and $u(y) = v(z)$, the equation becomes

$$z^2v''(z) + 2zv'(z) - \left(\frac{4\pi^2}{z^2}+\lambda\right)v(z) = 0,$$

where the coefficient of $v(z)$ is not analytic near 0. We call $y=\infty$ an *irregular singular point.* Dividing y^2 through the original equation, we have

$$u''(y) - \left(4\pi^2 + \frac{\lambda}{y^2}\right)u(y) = 0.$$

Remark A.0.5. Given an ODE of the following type

$$u''(y) + q(y)u(y) = 0$$

with an irregular singular point at ∞ such that $q(y)$ is both analytic at ∞ and not zero, the solution is aymptotic to the solution of

$$u''(y) + q(\infty)u(y) = 0$$

as $y \to \infty$, as is correctly suggested by the heuristic of *freezing* q at ∞. The justification can be seen in section 3.3 of [Erd10]. As noted at the end of section 3.2 of [Erd10], the proof mainly owes to [Hoh24] and [Tri53] (sections 47 to 50).

---

[12]An ordinary differential equation of the form $a(z)u''(z) + b(z)u'(z) + c(z)u(z) = 0$ has an *ordinary point* at z_0, provided $a(z_0) \neq 0$ (cf. [Ahl79]).

[13]An ordinary differential equation of the form $a(z)u''(z) + b(z)u'(z) + c(z)u(z) = 0$ has a *regular singular point* at z_0, provided $a(z_0) = 0$ (cf. [Ahl79]).

Freezing y at ∞ in our equation, we have

$$u''(y) - 4\pi^2 u(y) = 0,$$

which has well–known solutions $e^{2\pi y}$ and $e^{-2\pi y}$. We take $e^{-2\pi y}$ to buy the convergence at $y = \infty$. This gives the leading term of the asmptotic expansion of $u(y)$, which is thus of the form

$$u(y) \sim e^{-2\pi y} \sum_{m \geq 0} \frac{a_m}{y^m},$$ [14]

with $a_0 = 1$ by convention. Substituting this expression into the differential equation gives the recursion relation[15]:

$$a_m = \left(\frac{\lambda}{m} - (m-1) \right) \frac{a_{m-1}}{4\pi}.$$

Specifically,

$$\begin{aligned} a_1 &= \frac{\lambda}{4\pi} \\ a_2 &= \left(\frac{\lambda}{2} - 1 \right) \frac{\lambda}{(4\pi)^2} \\ a_3 &= \left(\frac{\lambda}{3} - 2 \right) \left(\frac{\lambda}{2} - 1 \right) \frac{\lambda}{(4\pi)^3} \\ &\cdots \\ a_m &= \left(\frac{\lambda}{m} - (m-1) \right) \cdots \left(\frac{\lambda}{2} - 1 \right) \frac{\lambda}{(4\pi)^m}. \end{aligned}$$

Since $y > 1$, we have

$$u(y) \sim e^{-2\pi y} \left(1 + O\left(\frac{\lambda}{y} \right) \right), \quad \text{as } y \to \infty.$$

It is important to note that this expansion is not necessarily uniform in λ, and we are particularly intereted in the case when $|\lambda|$ goes to infinity. Chapter 4 of [Erd10] mainly discusses the asymptotic solution of ODE's of the following type:

$$y'' + q(t, \lambda) y = 0.$$

In short, for t in a compact set and complex λ whose norm admits a lower bound and argument lies in a compact set, under some mild restrictions on $q(t, \lambda)$, the asymptotic solution has the same main term with that of $y'' + q(t)y = 0$, and has the same error term $O(\lambda^N)$ with $q(t, \lambda)$ as $|\lambda| \to \infty$, which is uniform both in t and the argument of λ. For our purpose, it suffices to take $t = y$. Therefore, the asymptotic expansion of $u(y)$ we obtained above is uniform for y in compact subsets of $(1, \infty)$ as $|\lambda| \to \infty$.

Acknowledgement

This note was written as my undergraduate thesis at the University of Minnesota, Twin Cities. I was introduced to the spectral theory of automorphic forms through Professor Paul Garrett's lectures in his seminar in Spring 2013. In addition, I was introduced to modern[16] analysis and its amazing applications through his graduate courses in the last two years. It is a pleasure to thank Professor Paul Garrett for his excellent guidance. Moreover, I would like to thank Professor Dennis Hejhal for helpful discussions.

---

[14] As noted in [Erd10], the idea of writing asymptotic solution in this form and obtaining the coefficients by recursion relation owes to G. D. Birkhoff and his pupils.

[15] Termwise differentiation in this context should be justified. Suppose f has an asymptotic power series at ∞. In fact, if f is differentiable and f' has an asymptotic power series at ∞, then the asymptotic expression of f' is obtained by differentiating that of f termwise. (cf. [Gar11a] [Erd10] [Olv74]).

[16] Post–L. Schwartz–I. Gelfand–A. Grothendieck (cf. [Gro53a] [Gro53b] [Gro55])–et al.

References

[Ahl79] L. Ahlfors, *Complex Analysis*, Third edition, International Series in Pure and Applied Mathematics (Book 7), McGraw-Hill, 1979.

[Bal67] C. B. Balogh, *Asymptotic expansions of the modified Bessel function of the third kind of imaginary order*, SIAM J. Appl. Math. **15** (1967), 1315–1323.

[Bar47] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. (2) **48**, No. 3 (1947), 568–640.

[BR99] J. Bernstein and A. Reznikov, *Analytic continuation of representations and estimates of automorphic forms*, Ann. of Math. (2) **150**, No. 1 (1999), 329–352.

[Bre11] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[CdV81] Y. Colin de Verdière, *Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein*, C. R. Acad. Sci. Paris Sér. I Math. **293**, No. 7 (1981), 361–363.

[CdV83] ———, *Pseudo-laplaciens, II*, Inst. Fourier (Grenoble) **33**, No. 2 (1983), 87–113.

[CM82] W. Casselman and D. Miličić, *Asymptotic behavior of matrix coefficients of admissible representations*, Duke J. Math. **49**, No. 4 (1982), 869–930.

[Erd10] A. Erdsélyi, *Asymptotic Expansions*, Dover Books on Mathematics, Dover Publications, 2010.

[Fri54] F. G. Friedlander, *Diffraction of pulses by a circular cylinder*, Comm. Pure Appl. Math. **2** (1954), 705–732.

[Gar05] P. Garrett, *Uniqueness of invariant distributions*, Retrieved July 2, 2013 from `http://www.math.umn.edu/~garrett/m/v/uniq_of_distns.pdf`, August 3, 2005.

[Gar10] ———, *Standard estimates for* $SL_2(\mathbb{Z}[i])\backslash SL_2(\mathbb{C})/SU(2)$, Retrieved September 8, 2013 from `http://www.math.umn.edu/~garrett/m/v/std_estimates.pdf`, June 6, 2010.

[Gar11a] ———, *Asymptotics at irregular singular points*, Retrieved April 20, 2013 from `http://www.math.umn.edu/~garrett/m/mfms/notes_c/irreg_sing_pt.pdf`, August 13, 2011.

[Gar11b] ———, *Global Automorphic Sobolev Spaces*, Retrieved May 14, 2013 from `http://www.math.umn.edu/~garrett/m/v/auto_sob.pdf`, January 6, 2011.

[Gar11c] ———, *The simplest Eisenstein series*, Retrieved August 24, 2013 from `http://www.math.umn.edu/~garrett/m/mfms/notes_c/simplest_eis.pdf`, June 7, 2011.

[Gar11d] ———, *Unbounded operators, Friedrichs' extension theorem*, Retrieved August 13, 2013 from `http://www.math.umn.edu/~garrett/m/v/friedrichs.pdf`, March 7, 2011.

[Gar11e] ———, *Asymptotics at regular singular points*, Retrieved April 20, 2013 from `http://www.math.umn.edu/~garrett/m/mfms/notes_c/reg_sing_pt.pdf`, May 7, 2011.

[Gar12a] ———, *Spectral Theory for* $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$, Retrieved August 24, 2013 from `http://www.math.umn.edu/~garrett/m/v/spec_th_quot.pdf`, January 4, 2012.

[Gar12b] ———, *Colin de Verdière's meromorphic continuation of Eisenstein series*, Retrieved August 13, 2013 from `http://www.math.umn.edu/~garrett/m/v/cdv_eis.pdf`, March 12, 2012.

[Gar12c] ———, *Vector-valued integrals*, Retrieved August 13, 2013 from `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/07e_vv_integrals.pdf`, November 11, 2012.

[Gar13] ———, *Functions on circle: Fourier series, I*, Retrieved June 26, 2013 from `http://www.math.umn.edu/~garrett/m/fun/notes_2012-13/04_blevi_sobolev.pdf`, April 3, 2013.

[Gel36] I. M. Gelfand, *Sur un lemme de la théorie des espaces linéaires*, Communications de l'Institut des Sciences Mathématiques et mécaniques de l'Université de Kharkoff et la Société Mathématique de Kharkoff, No. 4, vol. 13 (1936), 35–40.

[God66a] R. Godement, *Decomposition of* $L^2(\Gamma\backslash G)$ *for* $\Gamma = SL_2(\mathbb{Z})$, Proc. Symp. Pure Math. **9**, AMS (1966), 211–224.

[God66b] ———, *Spectral decomposition of cuspforms*, Proc. Symp. Pure Math. **9**, AMS (1966), 225–234.

[Gro53a] A. Grothendieck, *Sur certains espaces de fonctions holomorphes. I.*, J. Reine Angew. Math. **192** (1953), 35–64.

[Gro53b] ———, *Sur certains espaces de fonctions holomorphes. II.*, J. Reine Angew. Math. **192** (1953), 77–95.

[Gro55] ———, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Am. Math. Soc. **16** (1955).

[Has77] H. Hass, *Numerische Berechnung der Eigenwerte der Differentialgleichung* $y^2\Delta u + \lambda u = 0$ *für ein unendliches Gebiet im* $\mathbb{R}^2$, Diploma thesis, Institute of Applied Mathematics, University of Heidelberg (1977).

[HC54] Harish-Chandra, *The Plancherel formula for complex semisimple Lie groups*, Trans. AMS **76**, No. 3 (1954), 485–528.

[Hej81] D. A. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series*, Recent Progress in Analytic Number Theory, ed. H. Halberstam and C. Hooley, vol. 2, Academic Press, NY (1981), 95–110.

[Hej83] ———, *The Selberg Trace Formula for* $PSL(2,\mathbb{R})$*, Volume 2*, Springer Lecture Notes in Math. **1001**, 1983.

[HL16] G. H. Hardy and J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes*, Acta Math. **41**, Issue 1 (1916), 119–196.

[HL94] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegal zero (With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman)*, Ann. of Math. (2) **140**, No. 1 (1994), 161–181.

[Hoh24] G. K. H. Hoheisel, *Untersuchungen ber die asymptotische Integration linearer Differentialgleichungen*, J. Reine Angew. Math. **153** (1924), 228–244.

[IS95] H. Iwaniec and P. Sarnak, L^∞ *norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. (2) **141**, No. 2 (1995), 301–320.

[Iwa90] H. Iwaniec, *Small eigenvalues of Laplacian for* $\Gamma_0(N)$, Acta Arith. **16** (1990), 65–82.

[Iwa02] ———, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematis **53**, AMS, 2002.

[Lax02] P. D. Lax, *Functional Analysis*, Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts **55**, Wiley-Interscience, 2002.

[LP77] P. D. Lax and R. Phillips, *Scattering theory for automorphic functions*, Annals of Math. Studies (Book 87), Princeton University Press, 1977.

[MS04a] S. D. Miller and W. Schmid, *Distributions and analytic continuation of Dirichlet series*, J. Funct. Anal. **24**, No. 1 (2004), 155–220.

[MS04b] ———, *Summation formulas, from Poisson and Voronoi to the present*, Noncommutative harmonic analysis, Progr. Math. **220**, Birkhäuser Boston, Boston, MA (2004), 419–440.

[MS06] ———, *Automorphic distributions, L-functions, and Voronoi summation for* $GL_3(\mathbb{Z})$, Ann. of Math. (2) **164**, No. 2 (2006), 423–488.

[MS08] ———, *The Rankin-Selberg Method for Automorphic Distributions*, Representation Theory and Automorphic Forms Progress in Mathematics **255**, Birkhäuser Boston, Boston, MA (2008), 111–150.

[MS12] ———, *The Archimedean theory of the exterior square L-functions over* $\mathbb{Q}$, J. Amer. Math. Soc. **25**, No. 2 (2012), 465–506.

[Olv54] F. W. J. Olver, *The asymptotic solution of linear differential equations of the second order for large values of a parameter*, Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 307–327.

[Olv74] ———, *Asymptotics and special functions*, Academic Press, 1974.

[Pet38] B. J. Pettis, *On integration in vector spaces*, Trans. AMS **44**, No. 2 (1938), 277–304.

[Roe56] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1953/1955** (1956), 159–267.

[San04] P. Sanark, *Letter to C. Morawetz on bounds for eigenfunctions on symmetric spaces*, Retrieved Octobor 2, 2013 from `http://publications.ias.edu/sites/default/files/Sarnak_Letter_to_Morawetz.pdf`, August, 2004.

[Sel56] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

[Tri53] F. G. Tricomi, *Equazioni differenziali*, Second edition, Torino, 1953.