Partitions

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Introduction.

This paper is a summary of the research I did during my stay at the 2001 REU program in Number Theory at the University of Minnesota. As my topic, I chose partitions, and particularly the interplay of various fields of mathematics leading up to Rademacher's series[6] for the number of partitions of n.

Definition. *A partition of a number n is a set of positive integers whose sum is equal to n.*

Example: 3,2,2 (also denoted as 3+2+2) would be a partition of 7, as would 5+2 and 7 itself.

It is interesting to examine the number of possible partitions of n, which we will denote p(n).

Example: p(5)=7 because 5 can be partitioned as

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1,or 1+1+1+1+1

For convenience's sake, we will define p(0)=1 and p(n)=0 for $n < 0$ Other values of p(n) are listed in a table below (originally due to MacMahon)

n	p(n)
1	1
2	2
3	3
4	5
5	7
10	42
30	5,604
100	190,569,292
200	3,972,999,029,388

Generating Functions for Partitions.

Definition. *The generating function of a function f defined on the non-negative integers is defined as*

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n$$

.

Theorem 1. *The generating function for p(n) is $\prod_{n=1}^{\infty} \frac{1}{1-x^n}$. Alternatively,*

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} f(n)x^n$$

Proof: A formal derivation of the identity (ignoring questions of convergence) will be given here.

Formally,

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = (1+x+x^2+x^3+...)(1+x^2+x^4+x^6+...)(1+x^3+x^6+x^9+...)...$$

When written as a power series, the coefficient of the x^n term will correspond to the number of ways of getting x^n as a product of one term from each infinite sum, i.e. the number of solutions to

$$x^n = x^{1a_1}x^{2a_2}x^{3a_3}...$$

Taking the log of both sides to the base x,

$$n = 1a_1 + 2a_2 + 3a_3 + 4a_4 + ... = (1 + 1 + ... + 1) + (2 + 2 + ... + 2) + (3 + 3 + ... + 3) + ...$$

where the ith summand on the right-most side of the equation has a_i terms. But each right hand side is a partition of n and vice-versa, so the number of solutions (and therefore the coefficient of the power series) is p(n).

The proof above assumes we can blithely multiply together infinitely many power series without any difficulties cropping up, which may not necessarily be the case. For a more rigorous proof, see [1]. One thing which should be noted is that the product on the left (and therefore the sum on the right) converge iff $\sum_{n=0}^{\infty} x^n$ converges iff $|x| < 1$.

Euler's Pentagonal Number Theorem and a Recursive Formula for p(n).

By the exact same argument as for the generating function for p(n), we can create generating functions for other types of partitions. For example, $\prod_{n=1}^{\infty} \frac{1}{1-x^{2n}}$ counts the number of partitions of n into parts that are even, and $\prod_{n=1}^{\infty} \frac{1}{1-x^{n^2}}$ counts the number of partitions of n into parts that are square numbers. $\prod_{n=1}^{\infty}(1+x^n)$ is a slightly different matter; it only counts the partitions of n into distinct parts. What, then, will happen with $\prod_{n=1}^{\infty}(1-x^n)$? Again, only partitions into distinct parts will contribute to this sum, but this time they may contribute 1 or -1. The former happens when the partition is into an even number of distinct parts, while the latter occurs when the partition is into an odd number of distinct parts. Thus

$$\prod_{n=1}^{\infty}(1 - x^n) = p_e(n) - p_o(n)$$

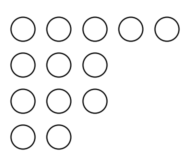
where $p_e(n)$ is the number of partitions of n into an even number of distinct parts and $p_o(n)$ is the number of partitions of n into an odd number of distinct parts. Interestingly enough, the difference on the right side is 0 except when n is a pentagonal number (a number of the form $\frac{3n(n-1)}{2}$) In fact,

Theorem 2 (Euler's Pentagonal Number theorem). *For* $|x| < 1$,

$$\prod_{n=1}^{\infty}(1 - x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - ... = 1 + \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}}$$

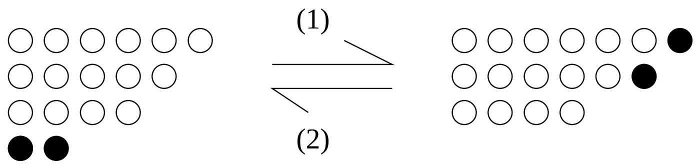
This theorem was originally proved by Euler using induction. Further proofs were obtained by Legendre in 1830 and by Jacobi (as a corollary of his triple-product identity) in 1846. Our proof is a combinatorial one due to Franklin[3].

Consider the graphical representation where the partition $a_1 + a_2 + ... + a_m$ is represented by a row of a_1 dots, a row of a_2 dots and so on down to a row of a_m dots. The partition 13=5+3+3+2 is represented below.



13=5+3+3+2+1■

To create a 1-1 correspondence between graphs and partitions, we require each row to be at least as long as the row after it. (For distinct partitions as in this theorem, each row will be strictly shorter than the one before it). Franklin's idea was to define two operations on the graphical representation of a partition that either add or subtract a single row (thus changing the parity of the number of partitions) and one operation is the inverse of the other. If exactly one of the operations is possible for each partition, we will



Operation 1 takes the left hand graph to the right. Operation 2 does the reverse.

have established a one-to-one correspondence between partitions into an odd number of distinct parts and those into an even number of distinct parts, so $p_e(n) = p_o(n)$

The two operations are as follows:

1. Take the shortest row of the graph, turn it 45 degrees, and place it to the right of the graph.

2. Take the 45-degree segment containing the upper right point of a graph and place it horizontally at the bottom row.

Operation 1 can be performed whenever the base is shorter than the 45 degree segment, or the base is equal to the 45 degree segment and they do not intersect.

Operation 2 can be performed whenever the 45 degree segment is shorter than the base, unless the difference between the two is exactly one and the two intersect.

Comparing the two cases, we see that it is never the case that both operations can be performed, and the only cases where neither operation can be performed are the two below.

The two cases where neither operation can be performed. The base and 45 degree segment intersect and either have the same length (left) or the base be one unit longer (right)

In the first case (if the base and 45 degree segment are both of length k),

$$n = k + (k+1) + (k+2) + ... + (2k-1) = \frac{(k)(3k-1)}{2}$$

In the second case (the base is length k+1, the 45 degree segment length k)

$$n = (k+1) + (k+2) + ... + (2k) = \frac{(3k+1)(k)}{2} = \frac{(-k)(3(-k)-1)}{2}$$

In either case, there is an excess odd partition if k is odd, an excess even partition if k is even, leading to the $(-1)^k$ factor as desired.

At this point we have

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \sum_{n=1}^{\infty} p(n)x^n$$

and

$$\prod_{k=1}^{\infty} (1-x^k) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n-1)}{2}}$$

Multiplying both equations together, we have

$$1 = (\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n-1)}{2}})(\sum_{n=1}^{\infty} p(n)x^n)$$

Equating coefficients of x^n, we get

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - ... = 0$$

(the sum on the left effectively terminates as soon as we get to p(k) for $k < 0$).

This equation can be used to find the values of p(n) assuming we already know the values of p(k) for $k < n$. This makes it useful to calculate p(n) for small n, but as we saw in our first table, p(n) grows rapidly enough to make this calculation difficult for larger n.

Asymptotic Growth Rates of p(n).

One of the most fundamental questions about partitions is their behavior for large n. Without too much effort we can make a few preliminary determinations about its growth.

Theorem 3. *$p(n) < p(n+1)$ for all n. p(n) is NOT $O(n^k)$ for any k. p(n) is $o(k^n)$ for all $k > 1$.*

Proof: First, we show that $p(n) < p(n+1)$. For any partition of n, create a partition of n+1 by affixing a +1 to the end. These new partitions are all distinct since the partitions of n were distinct, and there are p(n) of them. The strict inequality comes as we have not counted n itself as a partition yet.

Next, we obtain the (useful) result that p(1)+p(2)+...+p(n)<p(2n). Consider the following partitions of 2n.

$$2n = (2n-1) + (some\ partition\ of\ 1)$$
$$2n = (2n-2) + (some\ partition\ of\ 2)$$
$$.$$
$$.$$
$$2n = (2n-n) + (some\ partition\ of\ n)$$

The partitions in different rows are distinct has they have different largest members. We've counted p(1)+...+p(2n) partitions already, and some are still uncounted, so we have strict inequality.

Next, we show p(n) isn't $O(n^k)$. Assume by contradiction that it is, and let $C = \limsup_{n\to\infty} \frac{p(n)}{n^k}$ By definition of limsup, for every N and ϵ, we can find $n > N$ such that $p(n) > (C-\epsilon)x^k$. But then

$$p(4n) > p(1) + p(2) + p(3) + ... + p(2n) > p(n) + ... + p(2n) > n * (C-\epsilon)n^k$$

rearranging,

$$\frac{p(4n)}{(4n)^k} > (\frac{1}{4})^k (C-\epsilon)n > (\frac{1}{4})^k (C-\epsilon)N$$

The right side goes to infinity as N does, contradicting the limsup being finite. To see p(n) is $o(k^n)$, we need merely note that the generating function converges for $|x| < 1$, so $\sum_{n=1}^{\infty} p(n)(\frac{1}{k})^n$ converges, and the general term of that sum must go to 0.

In fact, Hardy and Ramanujan proved in 1918 that

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$$

This result can be obtained as a corollary of a result by Rademacher that gives p(n) explicitly as an infinite series. In order to understand that result, we need to venture a bit further afield into the area of modular forms.

Modular Forms.

Definition. *A Modular Form of weight k is an function analytic on the upper complex half-plane H (including at ∞) satisfying the relation*

$$f(\frac{at+b}{ct+d}) = (ct+d)^k f(t)$$

for all integers a,b,c,d with ad-bc=1.

(Here, analytic at infinity means that $\lim_{Im(z)\to\infty}$f(z) exists and is finite.) If a modular form is 0 at ∞, it is referred to as a cusp form. The transformations $t \to \frac{at+b}{ct+d}$ are called fractional linear transformations (or Möbius transformations). They act under composition just as the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

do under matrix multiplication. The group of 2x2 invertible matrices with integer entries is referred to as the modular group, SL_2, or just Γ. In practice, we will really only be working with SL_2 / $\{1, -1\}$, since the transformation represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the same as that represented by $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.

It should be noted that the sum of two modular forms of weight k is also a modular form of weight k, and that the product of modular forms of weight m and weight n is a modular form of weight m+n.

As it turns out, we do not need to check all of Γ to verify a function is a modular form. In fact, it suffices to have in place of all transformations just

$$f(t+1) = f(t)$$

and

$$f(-1/t) = t^k f(t).$$

Theorem 4. *The matrices S=*$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and T=*$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *generate* SL_2

(this is equivalent to saying every fractional linear transformation is some combination of $t \to t+1$ and $t \to \frac{-1}{t}$).

Proof: Define a Fundamental Domain of a group of transformations to be a set in the complex plane such that

(1) Every point in the upper half plane is taken to some point in the fundamental domain under some transformation.

(2) No two distinct points of the fundamental domain are taken to each other by any transformation in the group unless they are both on the boundary of the domain.

Our first goal will be to prove:

Lemma 1. *The Fundamental Domain of* Γ *is the set D=*$\{z \in H, |z| \geq 1 and |Re(z)| \leq \frac{1}{2}\}$

Proof: Take a point $z \in H$ and let Γ' be the group generated by S and T (which is certainly a subgroup of Γ at least). For any g in Γ, $Im(gz) = \frac{Im(z)}{|cz+d|^2}$. Since the lattice of points cz+d has a closest point to the origin, we can choose c and d such that Im(gz) is maximal. c and d are relatively prime (or else we could divide out by a common factor and get a closer point to the origin), so there exist a and b such that ad-bc=1. Take the element x of the modular group corresponding to that a,b,c,d, and let x' be the x composed with enough T's or T^{-1}'s so that $Re(x'(z)) < \frac{1}{2}$ (we can do this since T corresponds with the transformation $z \to z+1$. Note that Im(x'(z))=Im(z). x'(z) must be in D! Indeed, if it wasn't, $Sx'(z) = \frac{-1}{x'(z)}$ would have larger imaginary part, contradicting our choice of x. Therefore every z is taken to some point of D by an element of Γ.

Now suppose there are z and z' in D and g in Γ such that gz=z'. WLOG we can assume $Im(gz) \leq Im(z)$ (or else replace z by z', z' by z, and g by g^{-1}. This means $|cz+d| \leq 1$, so $Im(cz+d) = Im(cz) = cIm(z) \leq 1$. $Im(z) > \frac{1}{2}$ everywhere in D, so c must be -1, 0, or 1.

If c=0, d=1, a=1, and b is an integer, so we have the transformation $z \to z+b$. But $|b| \geq 1$(or else the transformation is the identity, and the width of D is 1, so this maps the interior of D off itself.

If c=1 or -1, $|cz+d| = |z+d| \geq |z| \geq 1$, with equality only on the boundary. Again, the point mapped into D could not have been in the interior of D originally.

Now we are ready to show Γ is indeed generated by S and T. Take any non-identity g in Γ and a point z_0 in the interior of D. $g(z_0)$ is not in D (by part 2 of the definition of Fundamental Domain). Therefore (by the first part of our proof of Lemma 1), there exists $h \in \Gamma'$ such that $h(g(z_0)) \in D$. But this can only happen if h(g(z)) is the identify, in which case $g = h^{-1}$, so $g \in \Gamma'$ and $\Gamma = \Gamma'$

Some examples.

Consider the series

$$G_k = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$

This series converges uniformly for $k \geq 3$, but for odd k it cancels out and becomes 0.

Theorem 5. *G_k is a modular form of weight k.*

Proof: As the uniform limit of functions analytic on H, G_k is analytic on H. As the imaginary part of z goes to infinity, all terms with $m > 0$ approach 0 and G_k becomes (effectively)

$$\sum_{n\neq 0} \frac{1}{n^k} = 2\zeta(k)$$

so G_k is also analytic at ∞.

$$G_k(z+1) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+(m+n))^k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k} = G_k(z)$$

$$G_k(\frac{-1}{z}) = \sum_{(m,n)\neq(0,0)} \frac{1}{(\frac{-m}{z}+n)^k} = z^k \sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^k} = z^k G_k(z)$$

Since G_k satisfies the modular equation for S and T, it does for all of Γ and we are done.

For convenience we define $g_4 = 60G_4$ and $g_6 = 140G_6$. $(g_4)^3$ and $(g_6)^2$ are both modular forms of weight 12, so

$$\Delta = (g_4)^3 - 27(g_6)^2$$

also is. At infinity, Δ takes the value

$$(120\zeta(4))^3 - 27(280\zeta(6))^2 = 0,$$

so Δ is actually a cusp form of weight 12.

Modular Forms of Certain Small Weights.

A useful formula regarding modular forms is as follows (see [7] for proof):

For any modular form of weight k not identically 0, let N be the number of 0's of the form in the closure of D, excluding the points i and $\rho = e^{2\pi i/3}$. Let N(i) and N(ρ be the order of the 0's (if any) at i and ρ respectively, and let N(∞) be the order of the 0 at ∞. We then have:

$$k = 12N + 6N(i) + 4N(\rho) + 12N(\infty)$$

As an example, we know Δ is a cusp form of weight k=12. Since $N(\infty) = 1$, all other terms on the right hand side must be 0. In particular, Δ has no finite 0's in H.

Using this formula, we can determine a few things about the modular forms of weight 0 and 12.

Theorem 6. *The only modular forms of weight 0 are the constant functions. The only cusp forms of weight 12 are multiples of Δ*

Proof: Let f be a modular form of weight 0, and let g(z)=f(z)-f(i). g has a 0 at i, to the right hand side of the equation above is positive, while the left hand side is 0. The only way out of this seeming condradiction is if g(z) is the 0 function, meaning f(z)=f(i) and f is constant.

In the case k=12, let f be any cusp form of weight 12, and consider g(z)=f(z)/$\Delta(z)$. g is analytic on H since Δ has no finite 0's. g is analytic at ∞ since both f and Δ have $N(\infty) = 1$. Therefore g is modular of weight 0, and must be constant, meaning f=cΔ.

A Connection between Modular Forms and Partitions: The Dedekind Eta Function.

Defintion. *Dedekind's Eta function is defined as:*

$$\eta(t) = e^{\pi it/12} \prod_{n=1}^{\infty} (1 - e^{2\pi int})$$

The product converges uniformly whenever $\sum e^{2\pi int}$ does, which is whenever $Im(t) > 0$. In this region, η is the uniform limit of analytic functions so is itself analytic.

If we let $x = e^{2\pi it}$, the infinite product can be written as $\prod_{n=1}^{\infty}(1 - x^n)$, which is just the reciprocal of the generating function for p(n). Consequently any information we get about the transformation properties of η can be used to derive similar properties for the generating function for partitions. A straightforward calculation gives

$$\eta(t+1) = e^{\pi i(t+1)/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in(t+1)}) = e^{\pi i/12}\eta(t),$$

which is almost but not quite the relationship required for a modular form.

Similarly, it can be shown (see [2] or Siegel's paper [8] for example) that

$$\eta(\frac{-1}{t}) = (-it)^{1/2}\eta(t)$$

(choosing the branch of the square root function that is positive for positive z). Taking this relation and the one above to the 24^{th}power and iterating the one above it 24 times, we see that.

$$\eta^{24}(t+1) = eta^{24}(t)$$

$$\eta^{24}(\frac{-1}{t}) = t^{12}eta^{24}(t)$$

η^{24} is analytic because η is. As $Im(t) \to \infty$, the infinite product in the definition of η^{24} goes to 1 (due to the uniform convergence of η and the fact that all partial products go to 1). Therefore

$$\lim_{Im(t)\to\infty} \eta^{24}(t) = \lim_{Im(t)\to\infty} e^{2\pi it} \prod_{n=1}^{\infty} (1 - e^{2\pi int})^{24} = \lim_{Im(t)\to\infty} e^{2\pi it} = 0,$$

so $\eta^{24}(t)$ is a cusp form of weight 12 and by theorem 6 must in fact be a constant multiple of $\Delta(t)$ (by comparison of Fourier Series it is possible to determine that the constant is $(2\pi)^{12}$, but we don't need that here).

We know from the definition of modular forms that

$$\Delta(\frac{at+b}{ct+d}) = (ct+d)^{12}\Delta(t)$$

Taking 24th roots of both sides and dividing out by a constant, we find

$$\eta(\frac{at+b}{ct+d}) = \epsilon(ct+d)^{1/2}\eta(t)$$

, where epsilon is a 24th root of 1. Determining which root ϵ is, however, is rather difficult. As it turns out (for a proof, see [2]),

$$\eta(\frac{at+b}{ct+d}) = exp(\pi i(\frac{a+d}{12c} + s(-d,c)))(-i(ct+d))^{1/2}\eta(t)$$

where

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}(\frac{hr}{k} - \lfloor\frac{hr}{k}\rfloor - \frac{1}{2})$$

Here $\lfloor x \rfloor$ represents the greatest integer $\leq$ x

A Functional equation for the Partition Generating Function.

Recall that

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n$$

Theorem 7. *Let $z \in H$ and choose h, k, and H so that $k > 0$, (h,k)=1, and $hH \equiv -1(modk)$. Let*

$$x = \exp(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}), x' = \exp(\frac{2\pi iH}{k} - \frac{2\pi}{z}).$$

Then

$$F(x) = e^{\pi is(h,k)}(\frac{z}{k})^{\frac{1}{2}} \exp(\frac{\pi}{12z} - \frac{\pi z}{12k^2})F(x')$$

Proof: Rearranging the equation for η slightly and letting t'=(at+b)/(ct+d), we find

$$\frac{1}{\eta(t)} = \frac{1}{\eta(t')} \exp(\pi i(\frac{a+d}{12c} + s(-d,c)))(-i(ct+d))^{1/2}$$

Since $F(e^{2\pi it}) = e^{\frac{\pi it}{12}}/\eta(t)$, we can plug in to get

$$F(e^{2\pi it}) = F(e^{2\pi it'}) \exp(\frac{\pi i(t-t')}{12})(-i(ct+d))^{\frac{1}{2}} \exp \pi i(\frac{a+d}{12c} + s(-d,c))$$

Choose $a = H, b = -\frac{hH+1}{k}, c = k, d = -h, t = \frac{iz+hk}{k^2}$. (b is an integer since $hH \equiv -1(modk)$. t is in the upper half plane since Re(z) and k are both > 0), and ad-bc=1. $t' = (at+b)/(ct+d) = \frac{H}{k} + \frac{i}{z}$, and plugging in the above equation gives the desired result.

Farey Fractions and Ford Circles.

Defition. *F_n, the set of Farey fractions of order n, is the set of all reduced fractions in [0,1] with denominators $\leq n$, listed in increasing order.*

Example: $F_7 = \{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\}$

Looking at F_7, we see that any two consecutive terms $\frac{a}{b}$ and $\frac{c}{d}$ satisfy bc-ad=1. This is not a coincidence, and, in fact,

Theorem 8. *Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with $0 \leq \frac{a}{b} \leq \frac{c}{d} \leq 1$ are consecutvie in some F_n if and only if bc-ad=1.*

Proof: First, assume bc-ad=1. $\frac{a}{b}$ and $\frac{c}{d}$ must be in lowest terms because a linear combination of a and b (and also one of c and d) gives 1. For $n \geq \max(b,d)$, they will be consecutive Farey Fractions unless some fraction $\frac{h}{k}$ satisfies $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$. If that were true, we would have $ck - dh \geq 1$ and $bh - ak \geq 1$. But we would then have

$$k = (bc-ad)k = b(ck-dh) + d(bh-ak) \geq b+d,$$

so (in fact) $\frac{a}{b}$ and $\frac{c}{d}$ must be consecutive in F_n for $\max(b,d) \leq n \leq b+d-1$. In fact, this bound is sharp, since we can slip $\frac{a+c}{b+d}$ in between the two fractions.

To show the other direction, we induct on n. The case n=1 is true by inspection. Now suppose that all consecutive fractions in F_{n-1} satisfy the relation, and $\frac{a}{b} < \frac{h}{k}$(assume WLOG $b < k$, since the proof is identical otherwise) are consecutive in F_n. If they are consecutive in F_{n-1}, we are done by the inductive hypothesis. If not, then we know (from the first part of our proof) that (with $\frac{c}{d}$ as the fraction originally next to $\frac{a}{b}$), k=b+d, and ck-dh=bh-ak1, so the induction is complete.

Farey Fractions can be represented geometrically as Ford Circles.

Definition. *The Ford Circle C(h,k) of a reduced fraction $\frac{h}{k}$ is the circle with radius $\frac{1}{2k^2}$ centered at $(\frac{h}{k}, \frac{1}{2k^2})$.*

Ford circles can be drawn either in the xy-plane or the complex plane. It can be seen from the defintion that all Ford Circles are tangent to the x axis (the real axis in the complex plane). In fact,

Theorem 9. *Two Ford circles C(a,b) and C(c,d) either are tangent or do not intersect at all. They are tangent if and only if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions.*

Proof:Let D be the distance between the centers of the two circles, and r and R be their radii. If the circles intersect at some point, the triangle inequality applied to the triangle consisting of the two centers and the intersection point would give $r + R \geq D$, with equality if and only if the triangle is degenerate, that is to say the circles are tangent. We have

$$D^2 = (\frac{a}{b} - \frac{c}{d})^2.$$

and also

$$(r+R)^2 = (\frac{1}{2b^2} + \frac{1}{2d^2})$$

$$\begin{aligned} D^2 - (r+R)^2 &= (\frac{a}{b} - \frac{c}{d})^2 + (\frac{1}{2b^2} - \frac{1}{2d^2})^2 - (\frac{1}{2b^2} + \frac{1}{2d^2})^2 \\ &= \frac{ad - bc}{bd}^2 - 4(\frac{1}{2b^2})(\frac{1}{2d^2}) \\ &= \frac{(ad-bc)^2 - 1}{b^2d^2} \end{aligned}$$

This is always at least 0 (since ad-bc is an integer not equal to 0), and equal to 0 only when $|ad - bc| = 1$, which is precisely the condition for consecutive Farey fractions. Therefore the circles are tangent if $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions, and non-intersecting otherwise, as desired.

Rademacher's Formula for p(n).

Theorem 10. *(Rademacher, 1937)*

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \frac{d}{dn} \frac{sinh(\frac{\pi}{k}\sqrt{\frac{2}{3}(n - \frac{1}{24})})}{\sqrt{n - \frac{1}{24}}}$$

where,

$$A_k(n) = \sum_{0 \leq h < k and (h,k)=1} e^{\pi i s(h,k) - 2\pi i n h/k}$$

and

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}(\frac{hr}{k} - \lfloor \frac{hr}{k} \rfloor - \frac{1}{2})$$

The basic tool for the proof of this theorem is the residue theorem of complex analysis.

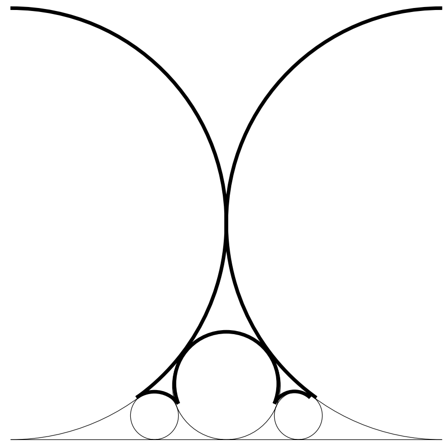
$$\frac{F(x)}{x^{n+1}} = p(0)x^{-(n+1)} + p(1)x^n + ... + p(n)x^{-1} + p(n+1) + ...,$$

so

$$\frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} = p(n),$$

where C is any simple closed contour containing 0 in the complex plane.

In particular, we choose C to be the circle of radius $e^{-2\pi}$ centered at the origin. Under the change of variable $x = e^{2\pi i t}$ (choosing the branch $0 \leq Re(t) < 1$), this contour becomes a horizontal segment between i and i+1 in the t-plane, while the region of convergence of F becomes the portion of H satisfying the inequality in Re(t).



Rademacher's path of integration in the case N=3.

The singularities in the x plane were at x=0 and along $|x| = 1$, so in the t plane they will be at infinity and at Im(t)=0. So long as we avoid enclosing singularities, we can replace our path from i to i+1 by any other path. Let us choose a path P(N) defined as follows: Starting at i, follow the Ford circle corresponding to 0 counterclockwise until the circle corresponding to the next fraction in F_n is encountered. Then follow that circle counterclockwise until the next one is encountered, and so on until the circle corresponding to 1 is traced to i+1. (The points of encounter are well defined since Ford circles of consecutive Farey fractions are tangent).

We now have

$$\int_C = \int_{P(N)} = \sum_{k=1}^{N} (\sum_{h<k,(h,k)=1} \int_{\gamma(h,k)})$$

where $\gamma(h,k)$ is an abbreviation for the upper arc of the Ford Circle C(h,k) traced during the integration.

From this point on the proof is primarily grunge and integral estimates, so I won't reproduce it here (see [2] for the full gory detail though). The basic steps are:

1. Use the transformation formula for F to replace the integrand by one in x' (as defined above).

2. Replace F(x') by ((F(x')-1)+1, and show that the integral of the term involving F(x')-1 tends to 0 as N tends to infinity.

3. Replace the integrals along $\gamma(h,k)$ by integrals along C(h,k) and show that the error incurred in doing so also tends to 0. (Even though C(h,k) is a closed contour, the integral around it is not 0 since F has singularities along the line t=0).

4. Let N (and therefore the number of circles we are integrating over) go to infinity.

5. Evaluate the integral on each C(h,k) in terms of Bessel Functions (specifically $I_{\frac{3}{2}}(z)$, and write the bessel functions as elementary functions (in this case, $I_{3/2}(z) = (\frac{2z}{\pi})^{\frac{1}{2}} \frac{d}{dz}(\frac{sinhz}{z})$ to get Rademacher's formula.

Rademacher's Formula and the Asymptotic Growth of p(n).

We can use Rademacher's Formula to verify Hardy and Ramanujan's asymptotic formula for p(n). By the triangle inequality, $|A_k(n)| < k^2$, so $|A_k(n)k^{1/2}| < \frac{5}{2}$ For large x, however, sinh(x) behaves in modulus like $e^x/2$, as does $\frac{d}{dx}sinh(x)$. The

$$\frac{d}{dn}\frac{sinh(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})})}{\sqrt{n-\frac{1}{24}}}$$

term, then, is asymptotic to

$$\frac{d}{dn}\frac{\exp(\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})})}{\sqrt{n-\frac{1}{24}}},$$

which is asymptotic to

$$\frac{d}{dn}\frac{\exp(\frac{\pi}{k}\sqrt{\frac{2}{3}(n)})}{\sqrt{n}}$$

The $\frac{1}{k}$ in the exponential will dominate the (at most $k^{\frac{5}{2}}$ term, so the infinite sum becomes dominated by the k=1 term and

$$p(n) \sim \frac{1}{2\pi\sqrt{2}}A_1(n)\frac{d}{dn}(\frac{\exp(\pi\sqrt{\frac{2}{3}n})}{2\sqrt{n}})$$

$A_1(n) = 1$ (h=0 is the only term in the sum, and s(0,1)=0), and (taking the derivative), this becomes

$$p(n) \sim \frac{1}{2\pi\sqrt{2}}\frac{e^{K\sqrt{n}}(-3+\pi\sqrt{6n})}{12n^3} \sim \frac{e^{K\sqrt{n}}}{4n\sqrt{3}},$$

where $K = \pi\sqrt{\frac{2}{3}}$. This is exactly Hardy and Ramanujan's formula.

Recent Developments: Congruences of the Partition Function.

Looking at Macmahon's table, Ramanujan noticed a pattern in the numbers
p(4)=5, p(9)=30, p(14)=135, p(19)=490,... and conjectured (and later proved)

$$p(5n+4) \equiv 0(mod5)$$

He later proved two other congruences

$$p(7n+5) \equiv 0(mod7)$$

$$p(11n+6) \equiv 0(mod11)$$

Apart from these, only a few other isolated examples were found over the next 80 years. A recent paper by K. Ono [4] has shown that (in fact) there are Ramanujan-style congruences for every prime modulus greater than 3.

It is still unknown whether any such congruences exist modulo 2 and modulo 3, but more work of Ono [5] shows that every arithmetic progression contains infinitely many n for which p(n) is even, and either infinitely many values for which p(n) is odd, or none of them.

The proof of both of these theorems rely on the properties of Modular Forms and the η function.

Conclusion.

As can be seen in this paper, the partition function can be approached in a variety of different ways. Combinatorics provides the idea of graphs of partitions and Franklin's proof of Euler's pentagonal number theorem. Number theory contributes the theory of modular forms and transformations for the η function. Analysis ties it all together with an integral formula to give us the asymptotic value of p(n). One function can tie together, in the end, a great deal of mathematics.

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References.

[1] Apostol, Tom M. *Introduction to Analytic Number Theory* Undergraduate Texts in Mathematics, New York-Heidelberg-Berlin:Springer-Verlag, 1976

[2] Apostol, Tom M. *Modular Functions and Dirichlet Series in Number Theory,* Graduate Texts in Mathematics 41, New York-Heidelberg-Berlin:Springer-Verlag, 1990

[3] Franklin, F. Sur le developpement du produit infini $(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots$ *Comptes Rendus Acad. Sci.* (Paris), 92,1881,448-450

[4] Ono, Ken., Distribution of the partition function modulo m. ıAnnals of Mathematics, 151, 2000, 293-307

[5] Ono, Ken., Parity of the partition function in arithmetic progressions, *J. reing angew. math.*, 472, 1996, 1-15

[6] Rademacher, Hans. On the partition function p(n). *Proc. London Math. Soc. (2) 43* (1937), 241-254.

[7] Serre, Jean-Pierre. *A Course in Arithmetic.* Graduate Texts in Mathematics 7. Springer-Verlag, New York-Heidelberg-Berlin, 1973

[8] Siegel, Carl Ludwig. A simple proof of $\eta(-1/\tau) = \eta(\tau)\sqrt{\frac{\tau}{i}}$. *Mathematika 1* (1954) 4. MR 16, 16b.