Hecke L-functions on Algebraic Number Fields

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Summer 2002

Introduction:

Hecke L-functions of idele class characters of number fields are a generalization of several major classes of classical functions, including the Riemann and Dedekind zeta functions and Dirichlet L-series. Originally conceived by E. Hecke, they were reinterpreted around 1950 by John Tate, who was able to reprove their analytic continuation and functional equation using adelic Fourier analysis. Although my studies this summer were focused on the details of Tate's proof, the emphasis here will be on the L-functions themselves, along with all the necessary preliminary information. In the first section, the places and completions of number fields are introduced, multiplicative characters of the completions are defined and classified, and L-functions of these local characters are defined. In the second, global L-functions of idele class characters are constructed in terms of the local L-functions, and Hecke L-functions are introduced, along with their connection to the Riemann and Dedekind zeta functions. In the final section, I include the theorem, proven by both Hecke and Tate, that Hecke L-functions admit meromorphic continuations to the entire complex plane. I then conclude by using this theorem to examine the analytic properties of the Riemann zeta function. Over the course of the paper, some degree of familiarity with algebraic number theory and local fields is assumed. My source throughout is D. Ramakrishnan and R. Valenza's book *Fourier Analysis on Number Fields.*

Local L-factors:

We begin by defining L-functions of multiplicative characters of local fields, which we will then use to construct the global L-functions that are to be our primary subject.

First, we need the following definitions:

Definition. *Let F be a field. A function $|\cdot| : F \to \mathbf{R}$ is a valuation if it satisfies the following conditions:*

(i) $|x| \geq 0$ *for all* $x \in F$, *and* $|x| = 0$ *iff* $x = 0$.

(ii) $|xy| = |x|\,|y|$.

(iii) There exists $c \in \mathbf{R}_+$ *such that for all* $x, y \in F$, $|x + y| \leq c \max\{|x|, |y|\}$.

If $|\cdot|$ satisfies (iii) with $c = 2$, it satisfies the triangle inequality, and we say that $|\cdot|$ is Archimedean. If $|\cdot|$ satisfies (iii) with $c = 1$, then it satisfies the ultrametric inequality $|x + y| \leq \max\{|x|, |y|\}$, and we say that it is non-Archimedean. We will also require a notion of equivalence of valutions.

Definition. *Two valuations $|\cdot|$ and $|\cdot|'$ are equivalent if there exists a positive real constant s such that $|\cdot|' = |\cdot|^s$. This defines an equivalence relation that partitions the set of valuations of F into equivalence classes. The equivalence classes of nontrivial valuations are referred to as places of F.*

Notice that if two valuations are equivalent, then they are either both Archimedean or both non-Archimedean.

By Ostrowski's Theorem, every valuation on $\mathbf{Q}$ is equivalent to $|\cdot|_\infty$, the usual absolute value, or to $|\cdot|_p$, a p-adic absolute value, for some rational prime p. In other words, the places of $\mathbf{Q}$ are in bijective correspondence with the set consisting of the rational primes and the "prime at infinity." It is clear, moreover, that the set of non-Archimedean (or finite) places are in bijective correspondence with the set of rational primes.

Now consider the case of an arbitrary algebraic number field K (i.e. K a finite extension of $\mathbf{Q}$). Any place ν of K defines, by restriction, a place u of $\mathbf{Q}$ in the sense that given a representative valuation $|\cdot|_\nu$ in the place ν, we may define a valuation $|\cdot|_u$ on $\mathbf{Q}$ by $|x|_u = |x|_\nu$ for all $x \in \mathbf{Q}$, which in turn belongs to some place u of $\mathbf{Q}$. Whenever places ν of K and u of $\mathbf{Q}$ satisfy the relation $|x|_\nu = |x|_u$ for all x $\in \mathbf{Q}$ we say that ν lies over u and write $\nu | u$.

For any field F, let F_ν denote the completion of F with respect to a representative valuation, say $|\cdot|_\nu$, in the place ν. In other words, let F_ν denote the smallest superset of F with the property that, with respect to the metric induced by $|\cdot|_\nu$, every Cauchy sequence of elements of F converges to an element of F_ν.

The following proposition will be useful later:

Proposition 1. *Let K be a finite extension of $\mathbf{Q}$, and let u be a place of $\mathbf{Q}$. The number of places ν of K lying above u is less than or equal to $[K : \mathbf{Q}]$, the degree of the extension $K/\mathbf{Q}$.*

Proof: In characteristic 0, every finite extension is separable. Hence $K/\mathbf{Q}$ is separable, and thus we know that $K = \mathbf{Q}(\alpha)$ for some α in the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$. Let $p(x)$ be the minimal polynomial of α over F, and let $n = deg(p(x))$. Suppose $p(x)$ factors over $\mathbf{Q}_u$ as

$$p(x) = \prod_{j=1}^{r} p_j(x),$$

$r \leq n$, with corresponding roots $\alpha_1, \alpha_2, \ldots, \alpha_r \in \overline{\mathbf{Q}_u}$. Naturally, $[K : \mathbf{Q}] = deg(p(x)) = n$. It suffices to show that the number of places ν of K lying over u is less than the number of embeddings of K into $\overline{\mathbf{Q}}_u$ induced by the assignments $\alpha \to \alpha_j$. The proof falls into two sections:

(i) To see that whenever $\nu | u$, $K_\nu = \mathbf{Q}_u(\alpha_j)$ for some j, consider the following diagram:

$$\begin{array}{ccc} \overline{\mathbf{Q}} & & \overline{K}_\nu \\ | & & | \\ K = \mathbf{Q}(\alpha) & \hookrightarrow & K_\nu \\ | & & | \\ \mathbf{Q} & \hookrightarrow & \mathbf{Q}_u \end{array}$$

Notice that K_ν contains $\mathbf{Q}_u(\beta)$, where β is the image of α under the central embedding. Of course, β must be a root of $p(x)$, say α_j. Since $\mathbf{Q}_u(\alpha_j)$ is a finite extension of $\mathbf{Q}_u$, it is also complete, and contains both $\mathbf{Q}$ and α_j, meaning that $\mathbf{Q}_u(\alpha_j) \supseteq K_\nu$. So $K_\nu = \mathbf{Q}_u(\alpha_j)$.

(ii) Any extension $\mathbf{Q}_u(\beta)$ may be regarded as a topological vector space isomorphic to $(\mathbf{Q}_u)^m$, where m is the degree of the minimal polynomial of β over $\mathbf{Q}_u$. $(\mathbf{Q}_u)^m$ is endowed with a norm derived from the absolute value $|\cdot|_u$ on $\mathbf{Q}_u$. Thus each of the extensions $\mathbf{Q}_u(\alpha_j)$ admits an absolute value $|\cdot|$ of this form (which must agree with $|\cdot|_u$ on $\mathbf{Q}_u$). When this absolute value is restricted to the image of $K = \mathbf{Q}(\alpha)$ under the embedding into $\mathbf{Q}_u(\alpha_j)$, it induces a valuation $|\cdot|_\nu$ on K and a corresponding place ν. Namely, for all $x \in K$, $|x|_\nu = |\sigma_j(x)|$, where σ_j denotes the relevant embedding. But then $|\cdot|_\nu$ agrees with $|\cdot|_u$ on $\mathbf{Q}$, since for all $x \in \mathbf{Q}$

$$|x|_\nu = |\sigma_j(x)| = |x| = |x|_u.$$

So ν lies over u. By part (i), every $\nu|u$ arises in this way. Thus the number of ν lying over u is less than or equal to the number of embeddings of K into $\overline{\mathbf{Q}}_u$ or, equivalently, the number of associations $\alpha \to \alpha_j$, which is less than or equal to $[K : \mathbf{Q}] = n$. ///

Once again, consider K_ν, the completion of an algebraic number field K with respect to a place ν. Since K is a finite extension of $\mathbf{Q}$, it must be the case that K_ν is a finite extension of $\mathbf{Q}_u$, where u is the place induced by the restriction of ν, as described above. As I noted earlier, we know all the possible places of $\mathbf{Q}$, and hence we also know all of the possible completions. From these we may determine the completions of K. We may treat the cases in which ν is Archimedean and non-Archimedean separately:

(i) ν Archimedean: Clearly ν must lie over an Archimedean place of $\mathbf{Q}$, but the only place of this kind is the one corresponding to the standard absolute value $|\cdot|_\infty$. So K_ν must be a finite extension of $\mathbf{Q}_\infty = \mathbf{R}$, and thus $K_\nu = \mathbf{R}$ or $\mathbf{C}$.

(ii) ν non-Archimedean: It must be the case that ν lies over some finite prime p. Hence K_ν is a finite extension of $\mathbf{Q}_p$, a p-adic field.

Now, before we may proceed, it is necessary to examine a few of the properties of non-Archimedean local fields. Suppose that K_ν is such a field. First, define the set of local integers $\mathcal{O}_\nu$ of K_ν to be $\mathcal{O}_\nu = \{x \in K_\nu : |x|_\nu \leq 1\}$ which, one can readily verify, is a subring of K_ν. The set of local units $\mathcal{O}_\nu^\times$, which forms a group under multiplication, is defined to be $\mathcal{O}_\nu^\times = \{x \in K_\nu : |x|_\nu = 1\}$. In addition, one can show that $\mathcal{O}_\nu$ contains a unique maximal ideal $\mathcal{P}_\nu = \mathcal{O}_\nu \setminus \mathcal{O}_\nu^\times = \pi_\nu \mathcal{O}_\nu$, where π_ν, the uniformizing parameter, is an element of $\mathcal{O}_\nu$ of greatest possible absolute value less than one. (Note: this description of π_ν is in fact a sensible one, since $\mathcal{O}_\nu$ can be shown to be a discrete valuation ring.)

It follows by maximality of $\mathcal{P}_\nu$ that the quotient ring $\mathcal{O}_\nu/\mathcal{P}_\nu$ is a field and, moreover, it is of finite order. Let $q_\nu = |\mathcal{O}_\nu/\mathcal{P}_\nu|$. In general, when we take a representative absolute value $|\cdot|_\nu$ from a non-Archimedean place ν, we normalize it so that $|\pi_\nu|_\nu = q_\nu^{-1}$.

It is perhaps useful to consider the non-Archimedean local fields of the form $\mathbf{Q}_p$, the p-adic fields. In any $\mathbf{Q}_p$, the local ring of integers is $\mathbf{Z}_p$, the p-adic integers, the maximal ideal in each $\mathbf{Z}_p$ is $p\mathbf{Z}_p$ (hence the uniformizing parameter is p), and the order of the residue field is $q_p = |\mathbf{Z}_p/p\mathbf{Z}_p| = |\{\mathbf{Z}_p, 1+\mathbf{Z}_p, \dots, (p-1)+\mathbf{Z}_p\}| = p$. Using this result, we have:

Proposition 2. *Let K be an algebraic number field, let ν be a non-Archimedean place of K, and let p be the finite prime such that $\nu|p$. Then $q_\nu = |\mathcal{O}_\nu/\mathcal{P}_\nu| = p^{k_\nu}$ for some $k_\nu \geq 1$.*

Proof: Without too much difficulty, one can see that since K_ν is a finite extension of $\mathbf{Q}_p$, the residue field of K_ν, $\mathcal{O}_\nu/\mathcal{P}_\nu$, is a finite extension of the residue field of $\mathbf{Q}_p$, $\mathbf{Z}_p/p\mathbf{Z}_p$. Thus it must be the case that

$$\begin{aligned} q_\nu &= |\mathcal{O}_\nu/\mathcal{P}_\nu| = |\mathbf{Z}_p/p\mathbf{Z}_p|^{k_\nu} \qquad \text{(for some } k_\nu \geq 1) \\ &= p^{k_\nu} \qquad /// \end{aligned}$$

Characters:

Using our classification of the completions K_ν, we may now determine the forms taken by characters of the multiplicative subgroups of completions of each type, and ultimately define L-functions of each type of character. First, a few definitions:

Definition. *Let K_ν be as described above (not necessarily Archimedean or non-Archimedean). A character of K_ν^* (the set of nonzero elements of K_ν) is a (continuous) homomorphism from K_ν^* to $\mathbf{C}^*$. A character χ is said to be unitary if its codomain is S^1. We say that a character is unramified if its restriction to the group of local units, U_ν is trivial. Otherwise we say that it is ramified.*

Clearly, $K_\nu^* \simeq U_\nu \times \Gamma$, where $U_\nu = \{x \in K_\nu^* : |x|_\nu = 1\}$ and $\Gamma = \{y \in \mathbf{R}_+^* : y = |x|_\nu \text{ for some } x \in K_\nu^*\}$. Thus any character χ may be factored as $\chi = \mu|\cdot|^s$, where μ is the pullback of a unitary character on U_ν, uniquely determined by the restriction of χ, and $s \in \mathbf{C}$.

For each completion K_ν, the characters and corresponding L-functions are as follows:

(i) $K_\nu = \mathbf{R}$: The only unitary characters of $K_\nu^* = \mathbf{R}^*$ are the trivial character $\mu = 1$, which maps every element of $\mathbf{R}^*$ to 1, or the sign character $\mu = sgn$, which maps every $x \in \mathbf{R}^*$ to $\frac{x}{|x|}$. Given that any character χ factors as described above, we may define

$$L(\chi) = L(\mu|\cdot|^s) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & \mu = 1 \\ \pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}) & \mu = sgn \end{cases}$$

where, as usual, $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$.

(ii) $K_\nu = \mathbf{C}$: One can see, first, that the only unitary characters on $U_\nu = S^1$ are of the form $\mu_n : e^{i\theta} \mapsto e^{in\theta}$, for some $n \in \mathbf{Z}$. Thus the only continuous homomorphisms from $K_\nu^* = \mathbf{C}^*$ to itself are of the form

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$$

for $n \in \mathbf{Z}$ and $s \in \mathbf{C}$. Define

$$L(\chi_{s,n}) = (2\pi)^{-(s+\frac{|n|}{2})}\Gamma(s + \frac{|n|}{2})$$

(iii) K_ν is a finite extension of some $\mathbf{Q}_p$: In this case, for any character χ of K_ν^*, we simply define

$$L(\chi) = \begin{cases} \frac{1}{1-\chi(\pi_\nu)} & \text{if } \chi \text{ is unramified} \\ 1 & \text{otherwise.} \end{cases}$$

In any of the cases above, we define the L-function corresponding to a character χ to be $L(s,\chi) = L(\chi|\cdot|^s)$ for all $s \in \mathbf{C}$. Naturally, the right hand side is given by the relevant formula outlined above.

Global L-functions:

Now that we have defined local L-functions, we may proceed to global considerations. Ultimately, we will construct global L-functions as the products of these local factors.

Let K be an algebraic number field. The *adele group* of K is defined to be the restricted direct product of the completions K_ν of K with respect to the local rings of integers, $\mathcal{O}_\nu$. In other words,

$$\mathbf{A}_K = \{(x_\nu) \in \prod_\nu K_\nu : x \in \mathcal{O}_\nu \text{ for all but finitely many places } \nu \text{ of } K\},$$

where the product is the set-theoretic one, taken over all places ν of K. Clearly, $\mathbf{A}_K$ forms a group under componentwise addition.

More important for our purposes, though, is the *idele group* $\mathbf{I}_K$ of K, defined to be the restricted direct product of the multiplicative subgroups K_ν^* of the completions of K with respect to the local unit groups $\mathcal{O}_\nu^\times$, i.e.

$$\mathbf{I}_K = \{(x_\nu) \in \prod_\nu K_\nu^* : x \in \mathcal{O}_\nu^\times \text{ for all but finitely many places } \nu \text{ of } K\}$$

It is clear that $\mathbf{I}_K$ is a group under componentwise multiplication. In addition, there is an algebraic embedding

$$\begin{array}{ccc} K^* & \hookrightarrow & \mathbf{I}_K \\ x & \mapsto & (x, x, x, ...). \end{array}$$

This embedding is well defined, since K^* embeds in each K_ν^*, and for every $x \in K^*$, $|x|_\nu = 1$ for almost all places ν of K. We may define a norm $|\cdot|$ on the adeles (and, by restriction, on the ideles) as follows: for any $(x_\nu) \in \mathbf{A}_K$, let $|(x_\nu)| = \prod_\nu |x_\nu|_\nu$.

As before, we are interested in the characters of the idele group of K. In particular, we will be concerned with the idele class characters:

Definition. *An* <u>*idele class character*</u> *is a (continuous) homomorphism from* $\mathbf{I}_K$ *to* $\mathbf{C}^*$ *that is trivial on the image of* K^* *in* $\mathbf{I}_K$ *under the diagonal embedding mentioned above.*

Fix an idele class character χ. For each place ν of K, define a local character

$$\begin{array}{cccc} \chi_\nu : & K_\nu^* & \rightarrow & \mathbf{C}^* \\ & t & \mapsto & \chi(1, \ldots, t, 1, \ldots, 1) \\ & & & \uparrow \\ & & & \nu\text{th component} \end{array}$$

Then $\chi(y) = \prod_\nu \chi_\nu(y)$. It follows from a general result concerning the characters of restricted direct products that the χ_ν are trivial on $\mathcal{O}_\nu^\times$ (are unramified) for all but finitely many places ν. It is now possible to define the global L-function of such a character in terms of the local versions defined above.

Definition. *Let* χ *be an idele class character. The L-function of* χ *is defined to be*

$$L(\chi) = \prod_\nu L(\chi_\nu),$$

wherever this is convergent.

To ensure that this definition is a useful one, we must check that the product converges somewhere.

Theorem 1. *Let* χ *be an idele class character of exponent* $\sigma = Re(s) > 1$. *Then* $L(\chi) = \prod_\nu L(\chi_\nu)$ *is absolutely convergent and nonzero.*

Proof: We may write $\chi = \chi_0|\cdot|^s$, where χ_0 is unitary and $Re(s) > 1$. Recall that χ_ν is ramified for finitely many ν and, as such, $L(\chi_\nu) = 1$ at a finite number of places. As such, to establish the convergence of $L(\chi)$

we need only concern ourselves with the places at which χ_ν is unramified. By definition of $L(\chi_\nu)$, then,

$$\begin{aligned}\prod_\nu |L(\chi_\nu)| &= \prod_\nu |\frac{1}{1-\chi_\nu(\pi_\nu)}| \\ &= \prod_\nu |\frac{1}{1-\chi_{0,\nu}(\pi_\nu)|\pi_\nu|^s}| \\ &= \prod_\nu |\frac{1}{1-\chi_{0,\nu}(\pi_\nu)q_\nu^{-s}}|\end{aligned}$$

We must show that the logarithm of this product converges:

$$\begin{aligned}\log(\prod_\nu |L(\chi_\nu)|) &= \log(\prod_\nu \frac{1}{|1-\chi_{0,\nu}(\pi_\nu)q_\nu^{-s}|}) \\ &= \sum_\nu \log(\frac{1}{|1-\chi_{0,\nu}(\pi_\nu)q_\nu^{-s}|}) \\ &= \sum_\nu Re(\log(\frac{1}{1-\chi_{0,\nu}(\pi_\nu)q_\nu^{-s}})) \\ &= Re(\sum_\nu \log(\frac{1}{1-\chi_{0,\nu}(\pi_\nu)q_\nu^{-s}})\end{aligned}$$

Using the power series expansion of $\log(\frac{1}{1-x})$, we have

$$\log(\prod_\nu |L(\chi_\nu)|) = Re(\sum_\nu \sum_{m>0} \frac{\chi_{0,\nu}(\pi_\nu)^m q_\nu^{-ms}}{m})$$

Since $\chi_{0,\nu}$ is unitary and $Re(q_\nu^{-ms}) \leq |q_\nu^{-ms}| = q_\nu^{-m\sigma}$, it suffices to show that the sum $\sum_\nu \sum_{m>0} \frac{q_\nu^{-m\sigma}}{m}$ converges for all $\sigma > 1$. Each finite place ν of K lies above some finite place of $\mathbf{Q}$ corresponding to a rational prime p, and for any $\nu|p$, q_v is a positive power of p, by Proposition 2. In addition, the number of places ν of K lying above p is bounded above by $n = [K : F]$, by Proposition 1. We may rewrite the sum as

$$\begin{aligned}\sum_\nu \sum_{m>0} \frac{q_\nu^{-m\sigma}}{m} &= \sum_{\nu|p} \sum_p \sum_{m>0} \frac{q_\nu^{-m\sigma}}{m} \\ &= \sum_{\nu|p} \sum_p \sum_{m>0} \frac{p^{-mk_v\sigma}}{m} \qquad (k_v \geq 1) \\ &\leq \sum_{\nu|p} \sum_p \sum_{m>0} \frac{p^{-m\sigma}}{m} \\ &\leq n \sum_p \sum_{m>0} \frac{p^{-m\sigma}}{m} \\ &= n \log(\prod_p \frac{1}{1-p^{-\sigma}}) \\ &= n \log(\sum_{n\geq 1} \frac{1}{n^\sigma})\end{aligned}$$

But the sum $\sum_{n\geq 1} \frac{1}{n^\sigma}$ converges absolutely for $\sigma > 1$, so we have established the absolute convergence of $L(\chi)$. ///

So our definition was a reasonable one. Using this global L-function, we may make the following definition:

Definition. *Let χ be an idele class character. Define the corresponding Hecke L-function to be $L(s,\chi) = L(\chi|\cdot|^s)$, where the function on the right hand side is the global L-function defined above. For our purposes, it is convenient to define finite and infinite versions:*

$$L(s,\chi_f) = \prod_{\nu \text{ finite}} L(s,\chi_\nu)$$
$$L(s,\chi_\infty) = \prod_{\nu \text{ infinite}} L(s,\chi_\nu)$$

Naturally, $L(s,\chi) = L(s,\chi_f)L(s,\chi_\infty)$.

It is now possible to demonstrate the way in which the Riemann and Dedekind zeta functions arise as the finite parts of a Hecke L-function. Let $\chi = 1$, the trivial idele class character. Then each of the induced characters χ_ν are trivial for all places ν of K since, by definition,

$$\chi_\nu(t) = \chi(1,\ldots,1,t,1,\ldots) = 1$$

for all $t \in K_\nu^*$. Clearly, then, χ_ν is unramified as a character of K_ν^* for all places ν of K. Thus the finite part of the Hecke L-function is

$$\begin{aligned} L(s,\chi_f) &= \prod_{\nu \text{ finite}} L(s,\chi_\nu) \\ &= \prod_{\nu \text{ finite}} \frac{1}{1-\chi_\nu(\pi_\nu)|\pi_\nu|_v^s} \\ &= \prod_{\nu \text{ finite}} \frac{1}{1-|\pi_\nu|_v^s} \\ &= \prod_{\nu \text{ finite}} \frac{1}{1-q_\nu^{-s}} \end{aligned}$$

where we have used the fact that each χ_ν is trivial.

As we noted earlier, every finite place ν of $\mathbf{Q}$ corresponds to a rational prime p. Also, at the place p, the order of the corresponding residue field is $q_p = |\mathbf{Z}_p/p\mathbf{Z}_p| = p$. Hence, in this case,

$$\begin{aligned} L(s,\chi_f) &= \prod_p \frac{1}{1-q_p^{-s}} = \prod_p \frac{1}{1-p^{-s}}, \\ &= \zeta(s) \end{aligned}$$

the Riemann zeta function, for $Re(s) > 1$.

Now, with χ still the trivial idele class character, consider $L(s,\chi_f)$ over an arbitrary number field K. All of the non-Archimedean places ν of K correspond to prime ideals P_ν in the global ring of integers $\mathcal{O}_K$. Recall that the absolute norm of any ideal A in $\mathcal{O}_K$ is defined to be $N(A) = [\mathcal{O}_K : A]$. In particular, for any prime ideal P_ν, $N(P_\nu) = [\mathcal{O}_K : P_\nu]$ which, in fact, is equal to q_ν, the degree of the residue field $\mathcal{O}_K/P_\nu$. Hence, in this case,

$$\begin{aligned} L(s,\chi_f) &= \prod_{\nu \text{ finite}} \frac{1}{1-q_\nu^{-s}} \\ &= \prod_{P \text{ prime}} \frac{1}{1-N(P)^{-s}} \\ &= \sum_{A\neq\{0\}} \frac{1}{N(A)^s} \end{aligned}$$

where A ranges over the set of all nonzero ideals of $\mathcal{O}_K$. In this case, then, $L(s,\chi_f) = \zeta_K(s)$, the Dedekind zeta function of K, for $Re(s) > 1$.

Hecke L-Functions: Meromorphic Continuation and Functional Equation

For any character χ, let $\chi^\vee = \chi^{-1}|\cdot|$. We may now state the chief result proven in Tate's thesis.

Theorem 2. *Let χ be a unitary idele class character. Then $L(s,\chi)$, which is initially defined and holomorphic in $\{Re(s) > 1\}$ admits a meromorphic continuation to the whole s-plane, and satisfies the functional equation*

$$L(1-s,\chi^{\vee}) = \varepsilon(s,\chi)L(s,\chi)$$

where $\varepsilon(s,\chi) = \prod_{\nu} \varepsilon(\chi_{\nu}|\cdot|^{s}) \in \mathbf{C}^{*}$. Moreover, this meromorphic continuation is entire unless $\chi = |\cdot|^{-i\tau}, \tau \in \mathbf{R}$, in which case there exist poles at $s = i\tau$ and $s = 1+i\tau$ with residues $-Vol(C_K^1)$ and $|N(\mathcal{D}_{\mathrm{K}/\mathbf{Q}})|^{-\frac{1}{2}}Vol(C_K^1)$ respectively.

Tate's proof is too involved to be given here. In short, though, he realized the local L-factors as local "zeta integrals" of the form $Z(f,\chi) = \int_{K_{\nu}^{*}} f(x)\chi(x)d^{*}x$, where χ is a character of K_{ν}^{*} and f is a nice function on K_{ν}^{*}. The global L-functions are realized as global "zeta integrals" of the form $Z(f,\chi) = \int_{\mathbf{I}_K} f(x)\chi(x)d^{*}x$ where χ is an idele class character, and f is a nice function on the ideles of K. He was able to prove the meromorphic continuation and functional equations of the local and global zeta integrals and, using these results, to prove the theorem above. The calculation of the constants appearing in the formulas for the residues of $L(s,\chi)$ is extremely complicated and will not be presented here either. Although the values are not critical for our purposes, one may say, for what it's worth, that C_K^1 is the norm one idele class group, the quotient of $\mathbf{I}_K^1 = \{(x_{\nu}) \in \mathbf{I}_K : |x| = 1\}$ by K^{*}, and the volume is taken with respect to the quotient measure on the idele class group $C_K = \mathbf{I}_K/K^{*}$. $\mathcal{D}_{\mathrm{K}/\mathbf{Q}}$ is the global different of $K/\mathbf{Q}$, which is defined to be the inverse fractional ideal of $\mathcal{D}_{\mathrm{K}/\mathbf{Q}}^{-1} = \{x \in K : tr_{K/\mathbf{Q}}(x\cdot\mathcal{O}_K) \subseteq \mathbf{Z}\}$.

We may now consider the theorem's implications for the analytic continuation of the Riemann zeta function.

Recall that, when $K = \mathbf{Q}$, we have $\zeta(s) = L(s,\chi_f)$, where χ is the trivial idele class character and $Re(s) > 1$. It is immediate from the theorem that $\zeta(s)$ admits a meromorphic continuation to the entire complex plane. The theorem also allows us to identify its poles. Clearly χ is of the form $|\cdot|^{-i\tau}$, with $\tau = 0$. We know then that the corresponding Hecke L-function $L(s,\chi)$ has simple poles at $s = 0$ and $s = 1$. Now, since the only infinite place of $\mathbf{Q}$ is the one represented by the usual absolute value, and since the completion of $\mathbf{Q}$ with respect to this place is $\mathbf{R}$, the character induced by χ at this infinite place, say χ_{ω} will be a character of $\mathbf{R}^{*}$. As we noted above, this induced character will be trivial. If we refer back to our definition of the L-functions of characters of $\mathbf{R}^{*}$, we see that

$$L(\chi_{\omega}) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$$

Given that

$$\begin{aligned} L(s,\chi) &= L(s,\chi_f)L(s,\chi_{\infty}) \\ &= \zeta(s)L(s,\chi_{\omega}) \\ &= \zeta(s)\pi^{\frac{s}{2}}\Gamma(\frac{s}{2}), \end{aligned}$$

We can see that the simple pole of $L(s,\chi)$ at $s = 0$ may be accounted for by the gamma function which, as we know, has a simple pole there. The gamma function is analytic at $s = 1$, though, so ζ must have a simple pole there. Finally, since the gamma function is nonzero everywhere, this must be the only pole of the analytic continuation of the Riemann ζ function.

While this result has been known for more than a century, it is merely the simplest example of the power of theorem. Since Hecke L-functions generalize a host of classical functions, including the Dedekind zeta function of an arbitrary number field and Dirichlet L-series, its true implications are far more profound.

Ramakrishnan, Dinakar and Robert Valenza. *Fourier Analysis on Number Fields.* New York: Springer-Verlag, 1999.