A pinching estimate for solutions of the linearized Ricci flow system on 3-manifolds

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1   Introduction

An important component of Hamilton's program for the Ricci flow on compact 3-manifolds is the classification of singularities which form under the flow for certain initial metrics. In particular, Type I singularities, where the evolving metrics have curvatures whose maximums are inversely proportional to the time to blow-up, are modelled on the 3-sphere and the cylinder $S^2 \times \mathbb{R}$ and their quotients. On the other hand, Type II singularities (the complementary case) are much more difficult to understand. Despite this, it is known from the work of Hamilton that their singularity models are stationary solutions to the Ricci flow. This uses several techniques, including Harnack inequalities of Li-Yau-Hamilton type, the strong maximum principle for systems, dimension reduction, and the study of the geometry at infinity of noncompact stationary solutions (see§§14-26 of [H2].) In terms of Hamilton's program, at least two obstacles remain: obtaining an injectivity radius estimate for Type II solutions and ruling out the so-called cigar soliton (the unique complete stationary solution on a surface with positive curvature) as the dimension reduction of a Type II singularity model.[1][2]

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[1]In fact, Hamilton and Yau have announced informally that these are the only two obstacles and that they both would follow from obtaining a suitable differential matrix Harnack inequality of Li-Yau-Hamilton type for arbitrary solutions of the Ricci flow on compact 3-manifolds.

[2]Added in proof: Very recently, Perelman [P1] has given a proof of the Little Loop Conjecture in all dimensions without curvature restriction. See also [P2] for some further developments.

On the other hand, it is also conjectured by Hamilton that Type II singularities are not generic. If this conjecture can be proven with some definition of generic which implies that for any compact 3-manifold the Ricci flow with suitable surgeries (see [H5] for how to perform surgeries) does not form Type II singularities, then there would be no need for obtaining an injectivity radius estimate for Type II solutions or ruling out the so-called cigar soliton. Partly for these reasons we are motivated to study the linearized Ricci flow. Given an initial metric on a 3-manifold and the corresponding solution to the Ricci flow, one would like to understand the behavior of solutions with nearby initial metrics. The linearized Ricci flow system is the pair of equations (4)-(5) which we consider below. In this paper we obtain an apriori estimate for arbitrary solutions to the linearized Ricci flow on compact 3-manifolds which we hope may be useful in its study. The inspiration for this estimate comes from the works of Hamilton (§10 of [H1] and §24 of [H2]) and Gursky [G].

Recall that if $\left(M^3, g\left(t\right)\right)$ is a solution to the Ricci flow on a compact 3-manifold with positive scalar curvature, then Hamilton obtained the following parabolic Bochner-type estimate

$$\frac{\partial}{\partial t}\left(\frac{|Rc|^2}{R^2}\right) \leq \Delta\left(\frac{|Rc|^2}{R^2}\right) + \frac{2}{R}\nabla R \cdot \nabla\left(\frac{|Rc|^2}{R^2}\right). \tag{1}$$

See Lemmas 10.5 (with $\gamma = 2$) and 10.6 in [H1] for the positive Ricci curvature case, and the equation for Y in the proof of Theorem 24.7 in [H2] for the more general positive scalar curvature case. A sharpened form of this estimate is the main estimate in showing that the normalized Ricci flow evolves a closed 3-manifold with positive Ricci curvature into a spherical space form; see [H1], Theorem 10.1, or [H3], Theorem 5.3 for a simpler proof. A further extension of this estimate is used to show that for a Type I singularity of the Ricci flow on a closed 3-manifold not diffeomorphic to a spherical space form, there exists a sequence of dilations about points and times approaching the singularity time that limits to a quotient of the cylinder $S^2 \times \mathbb{R}$; see Theorem 24.7, Corollary 24.8 and Theorem 26.5 in [H2].

On the other hand, also recall that Gursky (see [G]) proved that if $\left(M^4, g\right)$ is a closed, oriented, 4-manifold with positive scalar curvature such that

$$\Delta R = -4\left|W^+\right|^2 - 2\left|Rc - \frac{1}{4}Rg\right|^2 + \frac{1}{6}R^2 \tag{2}$$

and α is a self-dual harmonic 2-form, then

$$\Delta\left(\frac{|\alpha|^2}{R^2}\right) + \frac{2}{R}\nabla R \cdot \nabla\left(\frac{|\alpha|^2}{R^2}\right) \geq 0. \tag{3}$$

There is a formal similarity between these two Bochner formulas. Note that Kähler-Einstein surfaces satisfy (2). On the other hand, the Ricci flow is the parabolic version of the equation for Einstein metrics (the fixed points of the volume normalized Ricci flow are the Einstein metrics). Furthermore, if $\left(M^{2k}, g\right)$

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is a Kähler manifold and α is a J-invariant 2-form, then $h(X,Y) \doteq \alpha(X, JY)$ is a J-invariant symmetric 2-tensor and $(\Delta_d \alpha)(X, JY) = (\Delta_L h)(X, Y)$, where Δ_L is the Lichnerowicz laplacian (defined below). Thus if α is also harmonic, then $\Delta_L h = 0$. The parabolic version of this last equation is the Lichnerowicz laplacian heat equation.

The above considerations partly motivate us to study the analogue of estimates (1) and (3) in the context of the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{4}$$

coupled to the Lichnerowicz laplacian heat equation

$$\frac{\partial}{\partial t} h_{ij} = (\Delta_L h)_{ij} \doteq \Delta h_{ij} + 2R_{kijl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki} \tag{5}$$

for a symmetric 2-tensor h. This is the linearized Ricci flow system and arises from linearizing the Ricci flow using a version of DeTurck's trick (see §2). A differential Harnack inequality of Li-Yau-Hamilton type, patterned after Hamilton's trace inequality for the Ricci flow [H4] and Li-Yau's seminal inequality for the heat equation [LY], for this coupled system was proved by Hamilton and one of the authors [CH] and interpreted geometrically in terms of linearizing the Ricci flow by S.-C. Chu and one of the authors [CC]. A complex version of this inequality was proven by Ni and Tam [NT] and applied to the study of the Kähler-Ricci flow. The linearized Ricci flow has been studied by Guenther, Isenberg and Knopf [GIK] at flat solutions from the point of view of maximal regularity theory. Additional work on differential Harnack inequalities of Li-Yau-Hamilton type which appear related to the linearized Ricci flow are in [C] and [CK].[3]

Analogous to (1) and (3), we may consider the quantity $\frac{|h|^2}{R^2}$ for solutions to the Ricci flow on 3-manifolds with positive scalar curvature. This is a pointwise measure of the size of h relative to the scalar curvature. More generally, since $R_{\min}(t)$ is a nondecreasing function for solutions to the Ricci flow, we may replace R by $R + \rho$, where $\rho \in [0, \infty)$ is chosen so that $R + \rho > 0$ at $t = 0$.

Main Theorem. *Let $(M^3, g(t))$ be a solution to the Ricci flow on a closed 3-manifold on a time interval $[0, T)$ with $T < \infty$ and let $\rho \in [0, \infty)$ be such that $R_{\min}(0) > -\rho$. If the pair (g, h) is any solution to the linearized Ricci flow system (4)-(5), then there exists a constant $C < \infty$ such that*

$$\frac{\partial}{\partial t}\left(\frac{|h|^2}{(R+\rho)^2}\right) \leq \Delta\left(\frac{|h|^2}{(R+\rho)^2}\right) + \frac{2}{R+\rho}\nabla R \cdot \nabla\left(\frac{|h|^2}{(R+\rho)^2}\right) \tag{6}$$

$$+ 4C\rho\frac{|h|^2}{(R+\rho)^2}.$$

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[3]In the latter reference, a Li-Yau-Hamilton inequality for the Ricci flow is proved which generalizes Hamilton's matrix inequality and has some formal similarities with linear inequalities.

Consequently, by direct application of the maximum principle, the norm of the solution to the linearized Ricci flow equation is comparable to the scalar curvature plus a constant:

$$|h| \leq C\left(R + \rho\right)$$

on $M \times [0, T)$, where C depends only on $g\left(0\right)$, ρ and T. Furthermore, when $\rho = 0$, C is independent of T.

Taking $\rho = 0$ and $h_{ij} = R_{ij}$, we obtain:

Corollary 1 *(Hamilton, [H1]) If $\left(M^3, g\left(t\right)\right)$, $t \in [0, T)$, $T < \infty$ is a solution to the Ricci flow on a closed 3-manifold with positive scalar curvature, then there exists a constant $C < \infty$ such that*

$$\frac{|Rc|}{R} \leq C$$

on $M \times [0, T)$.

In §2 we recall how the system (4)-(5) is obtained by linearizing the Ricci flow using a version of DeTurck's trick with a time-dependent background metric. In §3 we give the proof of equation (6), from which the main theorem follows. This depends on the nonnegativity of a certain degree 4 polynomial in 6 variables (Lemma 5), which is proved in §4.

2   The linearized Ricci flow system

This section is mainly to motivate our study of the linearized Ricci flow system. The reader well familiar with DeTurck's trick [D] may skip this section.

As we stated in the introduction, a solution to the *linearized Ricci flow system* consists of a complete solution $\left(M^n, g_o\left(t\right)\right)$, $t \in [0, T)$, to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{7}$$

coupled with a solution $h\left(t\right)$, $t \in [0, T)$, to the *Lichnerowicz laplacian heat equation*

$$\frac{\partial}{\partial t} h_{ij} = \left(\Delta_L h\right)_{ij} \tag{8}$$

where

$$\left(\Delta_L h\right)_{ij} \doteqdot \Delta h_{ij} + 2R_{kijl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki}.$$

The Lichnerowicz laplacian Δ_L is defined using the evolving metric $g_o\left(t\right)$. Our main interest is when the solution is compact. However, in view of compactness arguments in the category of pointed solutions, it may be of interest to study the linearized Ricci flow system for complete, noncompact solutions.

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This system arises as follows. Given a solution $(M^n, g_o(t))$, $t \in [0, T_o)$, to the Ricci flow, consider the *modified Ricci flow with time-dependent background metrics* $g_o(t)$:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i \tag{9}$$

where the 1-forms $W(t)$ are defined by

$$W(t)_\ell \doteq g(t)_{\ell k} \, g(t)^{ij} \left(\Gamma[g(t)]_{ij}^k - \Gamma[g_o(t)]_{ij}^k \right) \tag{10}$$

and the covariant derivatives are with respect to the metrics $g(t)$. This is De-Turck's trick [D] with a *time-dependent* background metric. Note that the solution $g_o(t)$ to the Ricci flow is itself also a solution the modified Ricci flow with background metrics $g_o(t)$.

There exists a unique solution to the initial value problem for the modified Ricci flow for short time. The *modified Ricci tensor*, which we define to be the rhs of (9), depends only on $g(t)$ and $g_o(t)$. We first compute the linearization of the modified Ricci flow about the solution $g_o(t)$. In this case we get the Lich-nerowicz laplacian heat equation. Hence the modified Ricci flow is a parabolic equation, which in turn, implies uniqueness and short time existence. DeTurck gave this argument as a new proof of the short time existence and uniqueness of solutions to the Ricci flow originally proved in [H1].

Let $\{g_{s,o}\}_{s \in (-\varepsilon, \varepsilon)}$ be a smooth, one-parameter family of initial metrics with $g_{0,o} = g_o(0)$. Consider the one-parameter family $\{g_s(t), t \in [0, T_s)\}_{s \in (-\varepsilon, \varepsilon)}$ of solutions to the modified Ricci flow:

$$\frac{\partial}{\partial t} (g_s)_{ij} = -2 (Rc_s)_{ij} + \nabla_i W_j + \nabla_j W_i \tag{11}$$

$$g_s(0) = g_{s,o}, \tag{12}$$

where

$$W_s(t)_\ell \doteq g_s(t)_{\ell k} \, g_s(t)^{ij} \left(\Gamma[g_s(t)]_{ij}^k - \Gamma[g_o(t)]_{ij}^k \right), \tag{13}$$

and the Ricci tensor and covariant derivative are with respect to the metrics $g_s(t)$. Recall that

$$g_0(t) \equiv g_o(t) \quad \text{and} \quad W_0(t) \equiv 0.$$

Define

$$v_{ij}(t) \doteq \left. \frac{\partial}{\partial s} \right|_{s=0} g_s(t)_{ij}. \tag{14}$$

We shall call v the *variation of the metric tensor*.

Let $V(t) = g_0(t)^{ij} v(t)_{ij}$. A standard computation yields (see for example [H1])

Lemma 2 *The* variation of the Ricci tensor *is*

$$\left. \frac{\partial}{\partial s} \right|_{s=0} \left(-2Rc[g_s(t)]_{ij} \right)$$
$$= \Delta v_{ij} + 2R_{kijl}v_{kl} - R_{ik}v_{kj} - R_{jk}v_{ki} + \nabla_i \nabla_j V - \nabla_i \nabla^k v_{kj} - \nabla_j \nabla^k v_{ki}$$
$$= (\Delta_L v)_{ij} + \nabla_i \nabla_j V - \nabla_i \nabla^k v_{kj} - \nabla_j \nabla^k v_{ki} \tag{15}$$

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Recall the algebraic *Einstein operator*

$$G\left(v\right)_{ij} \doteqdot v_{ij} - \frac{1}{2}Vg_{ij}$$

(which takes the Ricci tensor to the Einstein tensor: $G\left(Rc\right)_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$) and the *divergence*

$$\delta : C^{\infty}\left(S^2 T^* M\right) \to C^{\infty}\left(T^* M\right)$$

where

$$\delta\left(T\right)_i \doteqdot -g^{jk}\nabla_k T_{ij}.$$

The L^2 *adjoint* of δ

$$\delta^* : C^{\infty}\left(T^* M\right) \to C^{\infty}\left(S^2 T^* M\right)$$

is the same as the *Lie derivative operator* acting on the metric:

$$\delta^*\left(v\right)_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) = \frac{1}{2}\left(L_{v^*}g\right)_{ij}$$

where $\left(v^*\right)^i = g^{ij}v_j$. The last three terms on the rhs of equation (15) may be rewritten as

$$\nabla_i \nabla_j V - \nabla_i \nabla^k v_{kj} - \nabla_j \nabla^k v_{ki} = 2\left[\delta^*\left(\delta\left[G\left(v\right)\right]\right)\right]_{ij}.$$

Hence we have the following well-known identity:

Lemma 3 *The variation of the Ricci tensor has the form*

$$\left.\frac{\partial}{\partial s}\right|_{s=0}\left(-2Rc\left[g_s\left(t\right)\right]_{ij}\right) = \left(\Delta_L v\right)_{ij} + 2\left[\delta^*\left(\delta\left[G\left(v\right)\right]\right)\right]_{ij}.$$

We may rewrite the 1-form $W_s\left(t\right)$ as

$$W_s\left(t\right)_\ell = -\frac{1}{2}g_s\left(t\right)_{\ell k}g_0\left(t\right)^{kp}g_s\left(t\right)^{ij}\left(\nabla_i\left[g_0\right]_{jp} + \nabla_j\left[g_0\right]_{ip} - \nabla_p\left[g_0\right]_{ij}\right)$$

$$= W_s\left(t\right)_\ell = \frac{1}{2}g_s\left(t\right)_{\ell k}g_0\left(t\right)^{kp}\left(\delta\left[G\left(g_0\right)\right]\right)_p,$$

where the covariant derivatives are with respect to the metrics $g_s\left(t\right)$. Define

$$\left(g_0\right)^{-1} : C^{\infty}\left(T^* M\right) \to C^{\infty}\left(T^* M\right)$$

by

$$\left(g_0\right)^{-1}T \doteqdot g_s\left(t\right)_{\ell k}g_0\left(t\right)^{kp}T_p.$$

Then

$$W_s\left(t\right)_\ell = \frac{1}{2}\left[\left(g_0\right)^{-1}\left(\delta\left[G\left(g_0\right)\right]\right)\right]_\ell.$$

Hence the last two terms of the modified Ricci tensor can be expressed as

$$\nabla_i W_j + \nabla_j W_i = 2\left(\delta^* W\right)_{ij} = \left(\delta^*\left[\left(g_0\right)^{-1}\left(\delta\left[G\left(g_0\right)\right]\right)\right]\right)_{ij},$$

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so that

$$-2R_{ij} + \nabla_i W_j + \nabla_j W_i = -2R_{ij} + \left(\delta^* \left[(g_0)^{-1} \left(\delta \left[G\left(g_0 \right) \right] \right) \right] \right)_{ij}.$$

Thus the only change to DeTurck's modification of the Ricci flow that we have made is that we allow the background metric g_0 to depend on time. In particular, we take g_0 to be the solution of the Ricci flow that we are linearizing about.

The motivation for studying the linearized Ricci flow system is the following.

Proposition 4 *The variation $v(t)$ of the metric tensor $g(t)$ corresponding to a one-parameter family $\{g_s(t)\}_{s \in (-\varepsilon, \varepsilon)}$ of solutions to the modified Ricci flow (11)-(12) is a solution to the Lichnerowicz laplacian heat equation:*

$$\frac{\partial}{\partial t} v = \Delta_L v.$$

That is, the pair (g, v) is a solution to the linearized Ricci flow system (7)-(8).

Proof. We compute

$$
\begin{aligned}
\frac{\partial}{\partial t} v_{ij}(t) &= \frac{\partial}{\partial t} \left. \frac{\partial}{\partial s} \right|_{s=0} g_s(t)_{ij} = \left. \frac{\partial}{\partial s} \right|_{s=0} \frac{\partial}{\partial t} g_s(t)_{ij} \\
&= \left. \frac{\partial}{\partial s} \right|_{s=0} \left(-2R_{ij} + \nabla_i W_j + \nabla_j W_i \right) \\
&= \left. \frac{\partial}{\partial s} \right|_{s=0} \left(-2Rc \left[g_s(t) \right]_{ij} \right) + \nabla_i \left(\left. \frac{\partial}{\partial s} \right|_{s=0} W_s(t)_j \right) + \nabla_j \left(\left. \frac{\partial}{\partial s} \right|_{s=0} W_s(t)_i \right) \\
&= (\Delta_L v)_{ij} + \nabla_i \nabla_j V - \nabla_i \nabla^k v_{kj} - \nabla_j \nabla^k v_{ki} \\
&\quad + \nabla_i \left(\frac{1}{2} g_0(t)^{k\ell} \left(\nabla_k v_{\ell j} + \nabla_\ell v_{kj} - \nabla_j v_{k\ell} \right) \right) \\
&\quad + \nabla_j \left(\frac{1}{2} g_0(t)^{k\ell} \left(\nabla_k v_{\ell i} + \nabla_\ell v_{kj} - \nabla_i v_{k\ell} \right) \right) \\
&= (\Delta_L v)_{ij},
\end{aligned}
$$

where we used $W_0(t) \equiv 0$ to obtain the fourth equality, and the fact that $\nabla_i \nabla_j V = \nabla_j \nabla_i V$ for the last equality.

3   Proof of the pinching estimate

In this and the following section, we give the derivation of equation (6), which implies the main theorem. Using the Ricci flow equation, (5), and the standard

equation $\frac{\partial}{\partial t} R = \Delta R + 2\left|Rc\right|^2$, we compute

$$\frac{\partial}{\partial t}\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) = -\frac{2}{\left(R+\rho\right)^2}\frac{\partial}{\partial t}g_{ij} \cdot h_{ij}^2 + \frac{2}{\left(R+\rho\right)^2}\left(\frac{\partial}{\partial t}h\right) \cdot h - 2\frac{\left|h\right|^2}{\left(R+\rho\right)^3}\frac{\partial}{\partial t}R$$

$$= \frac{4Rc \cdot h^2}{\left(R+\rho\right)^2} + \frac{2}{\left(R+\rho\right)^2}\left(\Delta h_{ij} + 2R_{kijl}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki}\right)h_{ij}$$

$$- 2\frac{\left|h\right|^2}{\left(R+\rho\right)^3}\left(\Delta R + 2\left|Rc\right|^2\right)$$

$$= \Delta\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) - 2\frac{\left|\nabla h\right|^2}{\left(R+\rho\right)^2} - 6\frac{\left|h\right|^2}{\left(R+\rho\right)^4}\left|\nabla R\right|^2$$

$$+ \frac{8}{\left(R+\rho\right)^3}h\nabla_i R \cdot \nabla_i h + \frac{4}{\left(R+\rho\right)^2}R_{ijkl}h_{il}h_{jk} - \frac{4}{\left(R+\rho\right)^3}\left|h\right|^2\left|Rc\right|^2 .$$

Since

$$\frac{1}{R+\rho}\nabla R \cdot \nabla\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) = -2\frac{\left|h\right|^2}{\left(R+\rho\right)^4}\left|\nabla R\right|^2 + \frac{2}{\left(R+\rho\right)^3}h\nabla_i R \cdot \nabla_i h,$$

we may rewrite the above evolution equation as

$$\frac{\partial}{\partial t}\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) = \Delta\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) + \frac{2}{R+\rho}\nabla R \cdot \nabla\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) \qquad (16)$$

$$- 2\frac{\left|\left(R+\rho\right)\nabla_i h_{jk} - \nabla_i Rh_{jk}\right|^2}{\left(R+\rho\right)^4} + \frac{4}{\left(R+\rho\right)^2}R_{ijkl}h_{il}h_{jk}$$

$$- \frac{4}{\left(R+\rho\right)^3}\left|h\right|^2\left|Rc\right|^2 .$$

When $n = 3$, we have the identity

$$R_{ijkl} = R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} - \frac{1}{2}R\left(g_{il}\,g_{jk} - g_{ik}\,g_{jl}\right).$$

Hence

$$\frac{\partial}{\partial t}\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) = \Delta\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right) + \frac{2}{R+\rho}\nabla R \cdot \nabla\left(\frac{\left|h\right|^2}{\left(R+\rho\right)^2}\right)$$

$$- \frac{2}{\left(R+\rho\right)^4}\left|\left(R+\rho\right)\nabla_i h_{jk} - \nabla_i Rh_{jk}\right|^2 + 4P$$

where

$$P = \frac{1}{(R+\rho)^2} \left[R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} - R_{jl}g_{ik} - \frac{1}{2}R\left(g_{il}\,g_{jk} - g_{ik}\,g_{jl}\right) \right] h_{il}h_{jk}$$

$$- \frac{1}{(R+\rho)^3} \left|h\right|^2 \left|Rc\right|^2$$

$$= \frac{1}{(R+\rho)^3} \left[2\left(R+\rho\right) Rc \cdot hH - 2\left(R+\rho\right) Rc \cdot h^2 + \frac{R}{2}\left(R+\rho\right)\left(\left|h\right|^2 - H^2\right) - \left|h\right|^2 \left|Rc\right|^2 \right]$$

$$= \frac{1}{(R+\rho)^3} \left[2RRc \cdot hH - 2RRc \cdot h^2 + \frac{1}{2}R^2\left(\left|h\right|^2 - H^2\right) - \left|h\right|^2 \left|Rc\right|^2 \right]$$

$$+ \frac{\rho}{(R+\rho)^3} \left[2Rc \cdot hH - 2Rc \cdot h^2 + \frac{1}{2}R\left(\left|h\right|^2 - H^2\right) \right].$$

The main theorem is now a consequence of the following inequality, which we shall prove in the next section.

Lemma 5 *We have for any metric g and symmetric 2-tensor h, the inequality*

$$\left|h\right|^2 \left|Rc\right|^2 - 2RHRc \cdot h + 2RRc \cdot h^2 + \frac{1}{2}R^2\left(H^2 - \left|h\right|^2\right) \geq 0.$$

This is because, then

$$\frac{\partial}{\partial t}\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right) \leq \Delta\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right) + \frac{2}{R+\rho}\nabla R \cdot \nabla\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right)$$

$$+ 4\frac{\rho}{(R+\rho)^3}\left[2Rc \cdot hH - 2Rc \cdot h^2 + \frac{1}{2}R\left(\left|h\right|^2 - H^2\right)\right].$$

On the other hand, we have the estimate $\left|Rc\right| \leq C\left(R+\rho\right)$ (see Theorem 24.4 of [H2]), which implies

$$\frac{\rho}{(R+\rho)^3}\left[2Rc \cdot hH - 2Rc \cdot h^2 + \frac{1}{2}R\left(\left|h\right|^2 - H^2\right)\right] \leq C\rho\frac{\left|h\right|^2}{(R+\rho)^2}.$$

Hence

$$\frac{\partial}{\partial t}\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right) \leq \Delta\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right) + \frac{2}{R+\rho}\nabla R \cdot \nabla\left(\frac{\left|h\right|^2}{(R+\rho)^2}\right) + 4C\rho\frac{\left|h\right|^2}{(R+\rho)^2}.$$

If $t \in [0,T)$, then applying the maximum principle implies

$$\frac{\left|h\right|^2}{(R+\rho)^2}\left(t\right) \leq C_0 \exp\left(4C\rho T\right),$$

where $C_0 = \max_{t=0} \left|h\right|^2 / \left(R+\rho\right)^2$. **q.e.d.**

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4 Nonnegativity of a degree 4 homogeneous polynomial in 6 variables

Proof of Lemma 5. Since h is symmetric, we may assume h is diagonal. Let h_1, h_2, h_3 denote the eigenvalues of h and let $r_1 = R_{11}, r_2 = R_{22}, r_3 = R_{33}$ denote the diagonal entries of R_{ij}. Then

$$-R^3 P = |h|^2 |Rc|^2 - 2RHRc \cdot h + 2RRc \cdot h^2 + \frac{1}{2}R^2 \left(H^2 - |h|^2\right)$$

$$\geq Q \doteq \left(h_1^2 + h_2^2 + h_3^2\right)\left(r_1^2 + r_2^2 + r_3^2\right)$$
$$- 2\left(r_1 + r_2 + r_3\right)\left(h_1 + h_2 + h_3\right)\left(r_1 h_1 + r_2 h_2 + r_3 h_3\right)$$
$$+ 2\left(r_1 + r_2 + r_3\right)\left(r_1 h_1^2 + r_2 h_2^2 + r_3 h_3^2\right) + \left(r_1 + r_2 + r_3\right)^2 \left(h_1 h_2 + h_1 h_3 + h_2 h_3\right),$$

where we used the inequality (throwing away the off-diagonal entries of R_{ij})

$$|Rc|^2 \geq r_1^2 + r_2^2 + r_3^2.$$

We expand and simplify this as

$$Q = r_1^2 h_1^2 + r_2^2 h_2^2 + r_3^2 h_3^2 + r_1^2 h_2^2 + r_1^2 h_3^2 + r_2^2 h_1^2 + r_2^2 h_3^2 + r_3^2 h_1^2 + r_3^2 h_2^2$$
$$+ r_3^2 h_1 h_2 + r_1^2 h_2 h_3 + r_2^2 h_1 h_3$$
$$- r_1^2 h_1 h_2 - r_1^2 h_1 h_3 - r_2^2 h_1 h_2 - r_2^2 h_2 h_3 - r_3^2 h_1 h_3 - r_3^2 h_2 h_3$$
$$- 2 r_1 r_2 h_1 h_2 - 2 r_1 r_3 h_1 h_3 - 2 r_2 r_3 h_2 h_3.$$

Writing Q as a bilinear form in $h = (h_1, h_2, h_3)$ with coefficients in $r = (r_1, r_2, r_3)$, we have

$$Q = \left(r_1^2 + r_2^2 + r_3^2\right) h_1^2 + \left(r_1^2 + r_2^2 + r_3^2\right) h_2^2 + \left(r_1^2 + r_2^2 + r_3^2\right) h_3^2$$
$$+ \left(-r_1^2 - r_2^2 + r_3^2 - 2 r_1 r_2\right) h_1 h_2 + \left(r_1^2 - r_2^2 - r_3^2 - 2 r_2 r_3\right) h_2 h_3$$
$$+ \left(-r_1^2 + r_2^2 - r_3^2 - 2 r_1 r_3\right) h_1 h_3.$$

The lemma follows from the claim that the polynomial Q takes nonnegative values for all real values of $r_1, r_2, r_3, h_1, h_2, h_3$. To prove the claim, it is enough to show that the Hessian matrix

$$H \quad := \quad \left[\frac{\partial^2 Q}{\partial h_i \partial h_j}\right]$$

$$= \quad \begin{bmatrix} 2r_1^2 + 2r_2^2 + 2r_3^2 & r_3^2 - r_1^2 - r_2^2 - 2\,r_1 r_2 & r_2^2 - r_1^2 - r_3^2 - 2\,r_1 r_3 \\ r_3^2 - r_1^2 - r_2^2 - 2\,r_1 r_2 & 2r_1^2 + 2r_2^2 + 2r_3^2 & r_1^2 - r_2^2 - r_3^2 - 2\,r_2 r_3 \\ r_2^2 - r_1^2 - r_3^2 - 2\,r_1 r_3 & r_1^2 - r_2^2 - r_3^2 - 2\,r_2 r_3 & 2r_1^2 + 2r_2^2 + 2r_3^2 \end{bmatrix}$$

is positive semidefinite for all real values of r_1, r_2, r_3. In turn, it is enough to show that the determinants

$$\Delta_1 := H_{11}, \quad \Delta_2 := \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix}, \quad \Delta_3 := \begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{vmatrix}$$

take nonnegative values for all real values of r_1, r_2, r_3. The nonnegativity of

$$\Delta_1 = 2r_1^2 + 2r_2^2 + 2r_3^2$$

is clear. The nonnegativity of

$$\Delta_2 = (2r_1 + 2r_2 + 2r_3)^2 - (r_1^2 - r_2^2 - 2r_2 r_3 - r_3^2)^2$$

follows from the inequality

$$2r_1^2 + 2r_2^2 + 2r_3^2 \geq r_1^2 + r_2^2 + 2|r_1 r_2| + r_3^2 \geq |r_1^2 - r_2^2 - 2r_2 r_3 - r_3^2|.$$

One has

$$3\Delta_3 = 2\,X^6 - 6\,X^4 Y - 24\,X^2 Y^2 + 24\,Y^3 + 16\,X^3 Z,$$

where

$$X := r_1 + r_2 + r_3, \quad Y := r_1^2 + r_2^2 + r_3^2, \quad Z := r_1^3 + r_2^3 + r_3^3.$$

The reader will have no difficulty verifying this identity with the help of a computer algebra package. Now on any circle in (r_1, r_2, r_3)-space defined by holding X and Y constant, the extremal values of Z occur at points with two coordinates equal, as one verifies by a straightforward Lagrange multiplier argument. Since for fixed X and Y, Δ_3 depends linearly on Z, it follows that on any circle defined by holding X and Y constant, the extreme values of Δ_3 are taken at points with two coordinates equal. After making the evident reductions, the inequality

$$\Delta_3(x, x, 1) = 4(8x^2 + 1)(x - 1)^2 \geq 0$$

suffices to prove the nonnegativity of Δ_3, and in turn the claim. With hindsight, the reader can see that it was overkill to actually write out the expression of Δ_3 in terms of X, Y and Z; all that was used in the proof of nonnegativity of Δ_3 was the fact that for fixed X and Y, Δ_3 depends linearly on Z. **q.e.d.**

References

[C]   B. Chow, *Interpolating between Li–Yau's and Hamilton's Harnack inequalities on a surface,* J. Partial Diff. Equations (China) **11** (1998) 137–140.

[CC] B. Chow and S.-C. Chu, *A geometric approach to the linear trace Harnack inequality for the Ricci flow,* Math. Res. Lett. **3** (1996) 549-568.

[CH] Chow, B. and R. S. Hamilton, *Constrained and linear Harnack inequalities for parabolic equations,* Invent. Math. **129** (1997) 213–238.

[CK] B. Chow and D. Knopf, *New Li-Yau-Hamilton inequalities for the Ricci flow via the space-time approach,* J. Differ. Geom. **60** (2002) 1–51.

[D]   D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Diff. Geom., **18** (1983) 157–162.; ibid. (improved version), in Selected papers on Ricci flow, edited by H.-D. Cao, etal., International Press, to appear.

[GIK] C. Guenther, J. Isenberg, and D. Knopf, *Stability of the Ricci flow at Ricci-flat metrics*, Comm. Anal. Geom. **10** (2002) 741–777.

[G]   M. Gursky, *The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics,* Ann. of Math. **148** (1998) 315-337.

[H1] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982) 255-306.

[H2] R. S. Hamilton, *The formation of singularities in the Ricci flow,* in Surveys in Differential Geometry **2** (1995) International Press, 7–136.

[H3] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986) 153-179.

[H4]   R. S. Hamilton, *The Harnack estimate for the Ricci flow*, J. Diff. Geom. **37** (1993) 225-243.

[H5]   R. S. Hamilton, *Four-manifolds with positive isotropic curvature,* Comm. Anal. Geom. **5** (1997) 1–92.

[LY]   P. Li and S. T. Yau, *On the parabolic kernel of the Shrödinger operator,* Acta Math. **156** (1986) 153–201.

[NT]   L. Ni and L.-F. Tam, *Kähler-Ricci flow and the Poincaré-Lelong equation,* Comm. Anal. Geom., to appear.

[P1]   G. Perelman, *The entropy formula for the Ricci flow and its geometric applications,* arXiv:math/0211159.

[P2]   G. Perelman, *Ricci flow with surgery on three-manifolds,* arXiv:math/0303109.