Finite Hecke Algebras and their Characters

(Say: The Hecke algebra is an important object in repn theory that in the affine case gives us info. about the repn theory of reductive p-adic groups. But we're going to talk today about the finite Hecke algebra. Let's start by giving three different definitions of the finite Hecke algebra and talking about why each one is important.

1) (Generators and Relations) (Deformation of Coxeter Gp. Alg.)

W: finite Coxeter gp., $W = \langle S \rangle$

$\mathcal{H}_W := \langle T_s \mid s \in S \rangle$

Braid: $\underbrace{T_s T_t \cdots}_{m_{st}} = \underbrace{T_t T_s \cdots}_{m_{st}}$

Quad: $T_s^2 = (q_s - 1) T_s + q_s$

(often, we take the q_s to all equal some cplx #. But here keep transcendental / $\mathbb{C}$)

$q_s = q_t \Leftrightarrow s \sim t$

For now, working over $\mathbb{C}[\{q_s\}]$

Basis: $\{T_w \mid w \in W\}$

* q-analogue to W e.g. trivial character now = length function

2) Borel-biinvariant functions on ~~reductive~~ gps. finite Chevalley gp G.

$B \leftarrow M$:

$$\left\{\begin{matrix}\text{repns of } G \\ \text{w/ } B\text{-fixed vector}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{repns of} \\ \text{Hecke alg.}\end{matrix}\right\}$$

3) Type A: Centralizer of quantum gp (Jimbo, 1986)

V: std repn of $U_q(gl_n)$ $\quad n > k$

$H_{S_k} := \mathrm{End}_{U_q(gl_n)}(V^{\otimes k})$

(don't get Hecke alg in other types)

<u>Thm</u>: These three definitions are equivalent

<u>Character Theory</u>

(Like for any alg. ob., want to study char. theory. We'll see later an application of this to knot theory.)

The first and most important tool is Tits' Deformation Thm)

Tits' Deformation Thm: Let W be a finite Coxeter gp, H_W its Hecke algebra over a "large enough" field k.

Then $H_W \cong k[W]$, and H_W is semisimple

(What this means is that the repn theory of H and W is "the same". Explicit isom. do exist, but used less often than Tits' Deformation Thm).

How do we define a character table for $\mathcal{H}$?

(Specifically, need to define "std" elts. on which we can take the characters and compute from them the char. values of the rest of the Hecke alg.)

Thm (Starkey, Ram, Geck-Pfeiffer): If λ is a CC class of W, we can take the std. elt. corresp. to λ to be T_{w_λ} for any min'l length $w_\lambda \in \lambda$.
(weighted ortog. rel'ns)

Computing the Character Table

1) "Inductive on Rank": M–N rule (types A, B, D, Ariki-Koike)
2) "By deformation": Starkey's Rule (type A)

Starkey's Rule (1975):

$$\chi(T_{w_\lambda}) = \sum_{\nu \vdash n} \overline{\chi}(w_\nu)\, p_\lambda^\nu \quad \text{where}$$

$$p_\lambda^\nu = \frac{|C_\nu \cap S_\lambda|}{|S_\lambda|} \det(q \cdot \mathrm{id}_{V_\lambda} - \rho_\lambda(w_\nu))$$

Ex: $W = A_2 = S_3$ $\quad s_1$ — s_2

S_3	1	s_1	$s_1 s_2$
$\overline{\chi_{triv}}$	1	1	1
$\overline{\chi_{sgn}}$	1	-1	1
$\overline{\chi_{ref}}$	2	0	-1

$$\chi_{ref}(T_{s_1}) = \sum_{\nu \vdash 3} \overline{\chi_{ref}}(w_\nu)\, p_{(21)}^\nu = 2p_{(21)}^{(1^3)} - p_{(21)}^{(3)}$$

$$p_{(21)}^{(1^3)} = \frac{1}{2}\det(q - \rho_{(21)}(w_{(1^3)})) = \frac{1}{2}(q-1)$$

$$p_{(21)}^{(3)} = 0$$

so $\chi_{ref}(T_{s_1}) = 2 \cdot \frac{1}{2}(q-1) = q - 1$

Application: Ocneanu's trace (used to construct HOMFLY poly) $\boxed{4}$

Starkey's Rule: computes the wts

$$\tau : H_W \to \mathbb{C}$$

$$\tau(h) = \sum_{\lambda \vdash n} \underbrace{a_\lambda}_{\text{wts.}} \chi_\lambda(h)$$

~~(These wts are in terms of Schur functions, so~~

(These wts. give positivity properties related to the classification of Von Neumann algebras).

(Type B, trace $\exists$, wts $\exists$, but proof uses type A wts; would be slicker pf + easier computationally to go directly there)

Starkey's Rule Proof

(One of my thesis problems is to develop a "Starkey's Rule" for type B)

Step	Extendability
1) $T_{w_0}^2$ central in H_W (Springer)	general
2) If $T_w^d = T_w^{2r}$, $\exists$ "deformation" formula for $\tau(T_w)$ (Broué-Michel)	general
3) Coxeter elts satisfy this property	all types, can extend (eg. longest elt)
4) Using ref'n repn, $\exists$ det formula for $\chi(T_w)$	types A & B* *new
5) Can use 4) to prove Starkey's Rule for any T_w w/ w: Coxeter elt. of std. parab. subgp	general, so works in types A & B
6) Every CC has such an elt	type A only

Strategies

1) Expand elts in step 3) (eg. "good" elts, quasiCoxeter elts)
2) Expand std. parabolic subgp. to other subgp. (nonstandard parabolic, other refl'n subgps)
3) Work backwards from Ocneanu's trace
4) Extend/modify char table construction

[illegible]