A Tour of the Representation Theory of $GL_2(\mathbb{Q}_p)$

① Rep theory definitions
② Motivation
③ Classification of reps & corresponding L-factors
④ Whittaker models & Casselman-Shalika formula

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① **Def** • A <u>representation</u> (π, V) of a group G is a homomorphism

$$\pi : G \longrightarrow GL(V), \quad V \text{ a } \mathbb{C}\text{-vector space}.$$

* If we pick a basis for V, $GL(V) \cong GL_n(\mathbb{C})$, $n = \dim V$
* Think of as an action of G on V
* If G has a topology, usually add more adjectives ("smooth reps")

• $\underline{\dim(\pi, V)} := \dim_{\mathbb{C}} V$

• A one-dim'l rep $\pi : G \longrightarrow \mathbb{C}^\times$ is called a <u>character</u>

• Given (π, V), the fnc. $\chi(g) = tr(\pi(g))$ is also called the <u>character of π</u>

Def A rep (π, V) is <u>irreducible</u> if the only subreps are $\{0\}$ and V.

Def (Induction) Given $H \leq G$ and a rep (χ, W) of H, we can define a representation $(\mathrm{Ind}_H^G \chi, V)$ of G, where

$$V = \{ f : G \longrightarrow W \mid f(hg) = \chi(h) f(g) \ \forall h \in H \}$$

$$G \curvearrowright V \text{ by } (s \cdot f)(g) = f(gs).$$

② Recall: Number Theorists ♡ automorphic forms ("generalization of periodic fn")
& L-functions — ("generating functions for arithmetic data")

$$G(\mathbb{A}) \curvearrowright \underbrace{\mathcal{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))}^{\text{space of a.f.ms}}$$
$G(\mathbb{A})$ acts by right mult

(for today $G := GL_2$)

$$\left(\mathbb{A} \text{ "=" } \prod_p' \mathbb{Q}_p \times \mathbb{R} \right)$$

so given $\phi \in \mathcal{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, get rep (π_ϕ, V_ϕ) where $V_\phi = G(\mathbb{A}) \cdot \phi$. These are called <u>automorphic representations</u>

Given π afc rep, have decomposition

$$\pi = \bigotimes_p \pi_p$$

into <u>local representations</u> of $G(\mathbb{Q}_p)$ (& $G(\mathbb{R})$)

* understanding local ireps helps us define local L-functions w/ the nice properties we like (analytic continuation, functional eqn) $L = \prod_p L_p$, $L_p = \prod \underbrace{L(s,\pi)}_{L\text{-factors for local reps}}$

③ Slogan (@least for $p \neq 2$): representations are parametrized by characters of tori $\cong (\mathbb{Q}_p^\times)^2$

Important subgroups of G: $\underset{\text{Borel}}{B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}$, $\underset{\text{unipotent}}{U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}}$, $\underset{\text{(split) torus}}{T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}}$, $B = TU$

$\boxed{\text{Classification}}$

ⓐ <u>1-dimensional reps</u>: $\chi \circ \det$ for χ a character of $\mathbb{Q}_p^\times$ (don't care too much about these)

ⓑ <u>Irreducible principal series</u> $\pi(\chi_1, \chi_2)$:
* take character of T, inflate to B, induce to G ("parabolic induction")

Explicitly: Let χ_1, χ_2 be chars of $\mathbb{Q}_p^\times$
- Define χ on T: $\chi\begin{pmatrix} a & \\ & b \end{pmatrix} = \chi_1(a)\chi_2(b)$
- Inflate to B by acting trivially on U: $\chi\left(\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \chi_1(a)\chi_2(b)$

· Induce to G: $\pi(\chi_1,\chi_2) = \mathrm{Ind}_B^G \chi$

If irreducible, define the L-factor:

$$L(s,\pi) = \frac{1}{(1-\chi_1(p)p^{-s})(1-\chi_2(p)p^{-s})}$$

(* χ_1, χ_2 unramified — otherwise replace $\chi_i(p)$ w/ 0)

These are irreducible, unless $\chi_1\chi_2^{-1} = |\cdot|_p^{\pm 1}$, in which case a subquotient is irreducible. These look like:

(c) $\pi(\chi|\cdot|_p^{1/2}, \chi|\cdot|_p^{-1/2})$, called <u>special reps</u>

$= St \otimes \chi$

$$L(s,\pi) = \frac{1}{1-\alpha p^{-s}}$$

where $\alpha = \chi(p)|p|_p^{1/2} = p^{-1/2}\chi(p)$ (or 0 if χ ramified)

(d) <u>Supercuspidal reps</u> — basically defined to be "all other reps". They are more complicated to describe.

The <u>Jaquet module</u> associated to a rep (π, V) is

$$J_V = V \big/ \langle \pi(u)v - v \mid u \in \mathcal{U}, v \in V \rangle$$

An irrep is <u>supercuspidal</u> if $J_V = 0$.

(Idea: Principal series — $\mathcal{U}$ acted trivially; if V supercuspidal, no elt of $\mathcal{U}$ acts trivially. Turns out these are the only options)

· obtained from chars on non-split tori: $\begin{pmatrix} a & b \\ Db & a \end{pmatrix}$ where $\sqrt{D} \notin \mathbb{Q}_p$

$$L(s,\pi) = 1$$

<u>Local Langlands</u>: $\left\{\begin{matrix}\text{(smooth)} \\ \text{reps of} \\ GL_2(\mathbb{Q}_p)\end{matrix}\right\} \xleftrightarrow{\text{bijection}} \left\{\begin{matrix}\text{(2-D, semisimple)} \\ \text{reps of} \\ \text{Weil group} \\ (\sim \mathrm{Gal}(\underset{\uparrow}{L}/\mathbb{Q}_p))\end{matrix}\right\}$

quadratic ext

& this bijection respects L-factors

"completed L-fnc"

& ε-factors ← (come up in functional eqn $\Lambda(\pi,s) = \varepsilon(\pi,s,\psi)\cdot\Lambda(\pi^\vee,1-s)$)

* In the case of $GL_2(\mathbb{F}_p)$ or $GL_2(\mathbb{Q}_p)$, this bijection is described explicitly in the books of Piatetski-Shapiro and Bushnell & Henniart, respectively

In the bijection,

special & supercuspidal reps $\longleftrightarrow$ irreps of Weil gp

principal series $\longleftrightarrow$ reducible reps

(4) Whittaker models & Casselman-Shalika

Let ψ be a character of $\mathbb{Q}_p$. Define $\psi_{\mathcal{U}}: \mathcal{U} \to \mathbb{C}^\times$ by
$\psi_{\mathcal{U}}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \psi(x)$.

<u>Def</u> A <u>Whittaker model</u> of a rep (π, V) of G is (the image of) an embedding

$$V \hookrightarrow \mathrm{Ind}_{\mathcal{U}}^{G} \psi_{\mathcal{U}}$$

In other words, it is a space $W(\pi)$ of fncs

$$W : G \longrightarrow \mathbb{C} \quad \text{s.t.}$$
$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) W(g).$$

<u>Why do we care about Whittaker models??</u>

* $\mathrm{Ind}_{\mathcal{U}}^{G} \psi_{\mathcal{U}}$ is multiplicity free (if they exist, W. models are unique)
* Whittaker functions give "Fourier decompositions" of automorphic forms
* Useful in proving analytic continuation & fnc'l eqn for L-fncs (by equating L's w/ "zeta integrals")

· Casselman-Shalika formula:

$\pi(\chi_1,\chi_2)$ [χ_1, χ_2 unramified] admits a Whittaker model

C-S formula computes the (spherical) Whittaker fnc explicitly:

$$W_0\left(\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}\right) = \underset{\substack{\uparrow \\ \text{some stuff}}}{(*)} \cdot \underbrace{\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2}}$$

where $\alpha_1 = \chi_1(p)$, $\alpha_2 = \chi_2(p)$, $m \geq 0$

$= $ Schur poly $S_\lambda(\alpha_1,\alpha_2)$, $\lambda = m$

$=$ value of character of irrep of $GL_2(\mathbb{C})$ on $\begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}$

For more general p-adic groups G:

value of sph. Whit. fnc $=$ $(*)\cdot$ character of Langlands dual group $G^L(\mathbb{C})$

Some references:

· Bump, "Automorphic Forms and Representations"

· Piatetski-Shapiro, "Complex Representations of GL(2,K) for finite fields K"

· Bushnell & Henniart, "The Local Langlands Conjecture for GL(2)."

· Kimball Martin, Automorphic representations course notes

· Emily's talk from summer rep theory seminar — see Claire's website :)
(good summary of P-S)