# Casselman-Shalika Formula for $GL_2$:

Outline
1. Whittaker models/functionals
2. Spherical representations
3. Spherical Whittaker function & the C-S formula
4. Significance

0. Automorphic representations decompose

$$\pi \cong \otimes \pi_v$$

local reps such as/ $GL_2(\mathbb{Q}_p)$

automorphic rep

* Our formula lives here in the local theory

Notation
- $F$ non-Archimedean local field
- $\mathcal{O}$ ring of integers
- $\mathfrak{p}$ maximal ideal of $\mathcal{O}$
- $q = |\mathcal{O}/\mathfrak{p}|$
- $\varpi$ uniformizer

$$G = GL_2(F), \quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

- Fix nontrivial additive character $\psi : F \to \mathbb{C}^\times$.

1. Whittaker models/functionals

def: Let $(\pi, V)$ be any irrep of $G$. A <u>whittaker functional</u> is a linear functional $L : V \to \mathbb{C}$ s.t.

$$L\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\right) = \psi(x) L(v)$$

Frobenius reciprocity

Whittaker functionals $\longleftrightarrow$ G-mod homs $V \hookrightarrow$ Space of $\mathrm{Ind}_N^G \psi$ = functions $f : G \to \mathbb{C}^\times$ s.t. $f(ng) = \psi(n) f(g)$, $n \in N$

A whittaker model is the image of such a hom existence? All smooth $\infty$-dim'l irreps have a Whittaker model.

! not true for $GL_n$ in general

uniqueness? The space of Whittaker models is at most 1-dim'l

↳ has nice consequences — will play a role in proof of the C-S formula

2. Spherical Representations:

Iwasawa decomposition: $G$ has a maximal compact subgroup $K := GL_n(\mathcal{O})$ and $G = B \cdot K$.

def: An irreducible admissible rep is spherical if it contains a $K$-fixed vector.

[In $\pi \cong \otimes \pi_v$, almost all are spherical]

existence? Any $\infty$-dim'l spherical irreducible admissible rep is a nonramified principal series rep

↳ $\chi_1, \chi_2$ nonramified quasicharacters of $F$

1. Inflate: $\chi\begin{pmatrix} y_1 & * \\ 0 & y_2 \end{pmatrix} := \chi_1(y_1)\chi_2(y_2)$

2. Induce : $\pi(\chi_1, \chi_2) := \mathrm{Ind}_B^G \chi$

The reps irreducible after this are nonramified principal series.

Here,

$$\varphi_K(bk) := \underset{\text{modular quasichar}}{\delta^{1/2}}\chi(b)$$

is the normalized spherical vector in $\pi(\chi_1, \chi_2)$.

Uniqueness? The space of K-fixed vectors is ~~[illegible]~~ for the ~~[illegible]~~ nonramified principal series is one dim'l

3. C-S formula:

def: Let $(\pi(\chi_1,\chi_2), V)$ be an unramified Principal series rep w/ $\alpha_i = \chi_i(\varpi)$. The spherical Whittaker function is the spherical vector in the Whittaker model:

$$W_0(g) := \Lambda(\pi(g)\varphi_K)$$ ! not really a defn

$$\Lambda(f) = \int_F f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\overline{\psi(x)}\,dx$$

makes $W_0(1) = 1 - q^{-1}\alpha_1\alpha_2^{-1}$ — comes from Fourier expansion of Eisenstein series

Note: If we fix $g$. $W_0(g)$ is a holomorphic function of $\alpha_1$ and $\alpha_2$.

Goal: compute this.

B/c of transforming properties of the components of $W_0(g)$, we only need to compute on cosets

$$NZ\backslash G/K$$

this + Iwasawa decomp $\Rightarrow$ only need to find $W_0\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix}$ for $m \in \mathbb{Z}$.

The Cassleman-Shalika formula:

Schur poly, $S_m(\alpha_1,\alpha_2)$.

$$(1-q^{-1}\alpha_1\alpha_2^{-1})^{-1} W_0\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} q^{-m/2}\dfrac{\alpha_1^{m+1} - \alpha_2^{m+2}}{\alpha_1 - \alpha_2}, & m \geq 0 \\ 0, & m < 0. \end{cases}$$

Appears (in more generality) in papers of C-S in 1980, not first proven by them, but by Kato and Shintani in different settings

Sketch of Casselman's method:

i. Can "easily" get $W_0\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} = 0$, $m<0$.

ii. Uniqueness of Whittaker functionals helps $\Rightarrow$ $(1-q^{-1}\alpha_1\alpha_2^{-1})W_0(g)$ is inv under $\alpha_1 \leftrightarrow \alpha_2$

iii. Out of $\varphi_K$, make

$$F_m(g) = \int_{\mathcal{O}} \varphi_K\left(g\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix}\right)dx$$

is fixed by the Iwahori subgroup $K_0(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid c \equiv 0 \bmod \mathfrak{p} \right\}$.
Express $F_m$ in the Casselman basis $\{\varphi_0, \varphi_2\}$ of $V^{K_0(\mathfrak{p})}$.

iv. Turns out

$$W_0\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} = \int_F \underbrace{F_m\left[\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right]}_{\substack{\text{sub in } F_m \text{ in terms} \\ \text{of Casselman basis}}} \psi(-x)\, dx$$

get $W_0\begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} = C_1 q^{-m/2}\alpha_1^m + C_0 q^{-m/2}\alpha_2^m$

integrals involving $\varphi_0, \varphi_1$

v. Compute $C_0$ & apply FE from ii //

4. <u>Significance & History</u>:

- $S_m(\alpha_1, \alpha_2)$ is the value of a character of an irrep of $GL_2(\mathbb{C})$ (Lie gp) on the conjugacy class of $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$
  - ↳ $GL_2(\mathbb{C})$ is the "L-gp" or Langlands dual group of $GL_2$...
    another example: $Sp_4{}^\vee = SO_5$ for higher rank or other p-adic reductive gps
- More generally, values of spherical Whittaker functions of an irrep are given by a character of the "L-group" applied to the conjugacy class parameterizing it

<u>History</u>: The L-group was introduced by Langlands in 1967. He conjectured about the role of these L-groups in the subsequent years — including this formula.

↳ Corresponding formula for $GL_n$ proven by Shintani in 1976
More general reductive gps Kato in 1978.

- Not only significant for this connection — but also aids in calculations involving L-functions, in Rankin-Selberg method.
or
Langlands-Shahidi method.

Sources:


- "Automorphic Forms and Representations" Bump
- "The L-group" Casselman
- Katy's notes for nice background on rep theory of $GL_2(\mathbb{Q}_p)$.