① SNTS
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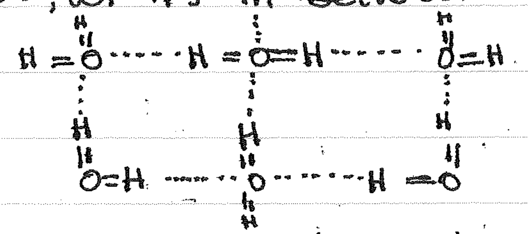
Intro to Ice Models in Number Theory

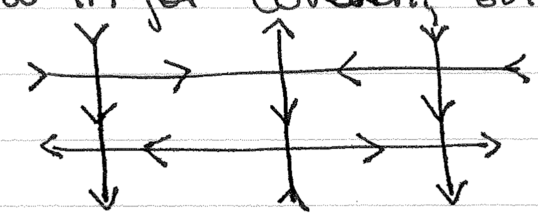
① What is an Ice Model?
② an absurd connection
③ My research

① What is an Ice Model?

- stolen from statistical mechanics
- first introduced 1935 Linus Pauling (residual entropy of water)
- models 2D sheet of ice, so single molecule-thick layer
  - H_2O, when freezes, snaps to square grid of O's, w/ H's in between



- each O has 2H conn. by covalent (strong) bonds & is conn. to 2H on other molecules by hydrogen (weak) bonds.
  - covalent ⇒ near ; hydrogen ⇒ far
- great, but involves a <u>lot</u> of lines ; let's simplify: place vertices for O's, edges btwn, and put arrow in for covalent, out for hydrogen

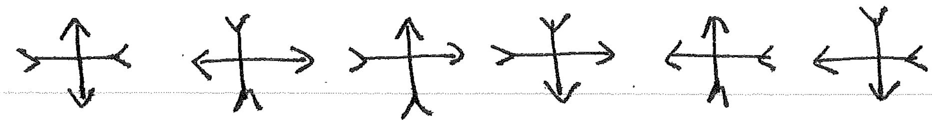


the Ice Model!

<u>Ice Model Rules</u>

- m×n grid of vertices w/ edges btwn; each vtx has 4 edges (tetravalent)
- each vertex must follow the ice rule: 2 in 2 out

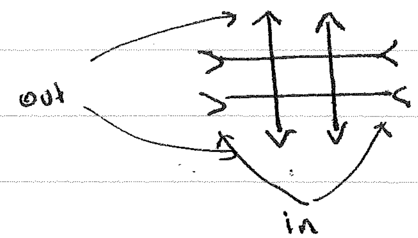
i.e. these are our 6 vertices:

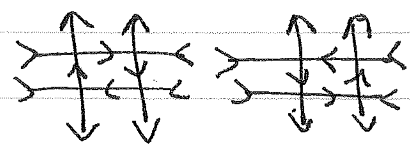
leave up  11/4/19

So it's the 6-vertex model, specifically.

Okay, now what?

- statistical mechanics ppl using it to look at behavior w/ a fixed set of bdry conditions
- ex: domain wall bdry conditions (DWBC)



fillings: 

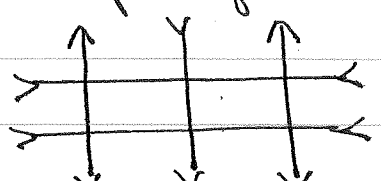
- let's define a slightly looser set:

Def: ~~let λ be~~ a <u>partition</u> is a sequence of nonincreasing positive integers: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$
i.e. weakly decreasing — finite for our purposes

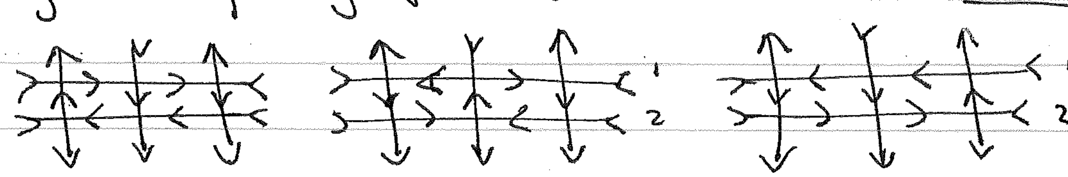
Ex: $(1,0,0)$, $(2,1)$, $(3,3,1,0)$, etc.

Def: let $\mathfrak{S}_\lambda$ be bdry conditions given by ptn λ in following way:
- add $\rho = (n-1, n-2, \ldots, 2, 1, 0)$ to λ to make $\lambda + \rho$ strictly dec.
- need at least $\lambda_1 + n$ col & n rows
- label columns in ice right to left; # rows top to bottom
- left, bottom, right bdries same as DWBC
- top edge: ^ if column appears in $\lambda+\rho$, v else.

System dep on λ

Ex:  $\lambda = (1,0)$ $\lambda + \rho = (2,0)$

- any valid (admissible) filling of arrows is called a <u>state</u>

* 

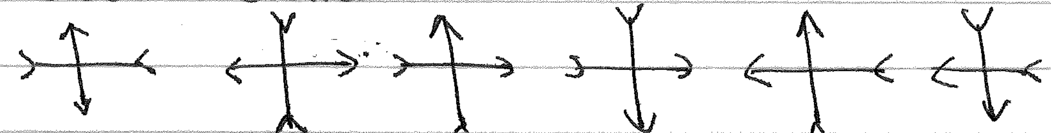
What now? Let's make fcns: define weighting system: assign wts to vertices based on type & row 11/4/19

Sometimes also column, but not here

- $\mathrm{wt}(\text{state}) = \prod_{\text{vertex} \in \text{state}} \mathrm{wt}(\text{vertex})$
- $\mathrm{wt}(\text{system}) = \sum_{\text{state} \in \text{system}} \mathrm{wt}(\text{state})$

very confusingly, called partition fcn $Z(\text{system})$.

② An absurd connection



Call based on arrows in

EW	NS	SW	NW	SE	NE
1	z_i	0	1	z_i	z_i

So going back to example

Ex 4: wt 0 wt $1 \cdot z_1 \cdot 1 \cdot 1 \cdot 1 \cdot z_2$ wt $1 \cdot z_1 \cdot z_1 \cdot 1 \cdot 1 \cdot 1$

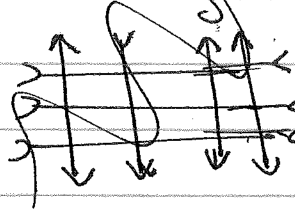
so $Z(\mathfrak{S}_{(1,0,0)}) = z_1 z_2 + z_1^2 = z_1 \underbrace{(z_1 + z_2)}_{\text{symmetric poly}}$

Thm: $Z(\mathfrak{S}_\lambda) = z^\rho s_\lambda(z_1, \dots, z_n)$

s_λ: Schur polynomial; super important symmetric polynomial in combo

~~Ex: $\lambda + \rho = (3,1,0)$ ($\lambda = (1,0,0)$)~~

~~only nonzero stts are~~



less than enlightening ex, actually.

Also, watching ppl do ice models on board is boring

This is nuts!

Question: if we pick different wts, what happens?

- Several weeks ago, Emily talked about Whittaker fcns $W_z(g)$, which are fcns we define on principal series reps of classical gps like $GL_n(F)$.

ϖ uniformizer of F

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Turns out: w/ $\overset{\text{dff set of}}{\text{weights}}$, $Z(\mathfrak{S}_\lambda) = \cancel{\ } z^{w_0\rho} W_z(\varpi^\lambda)$, where $\varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \varpi^{\lambda_n} \end{pmatrix}$

- we can also define Whittaker fcns on central covers of classical gps; i.e.

$$1 \longrightarrow \mu_n \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

- these covers, for $G = GL_n$, param. by bilin form B, so have to take that into account
- prin. series reps more complicated, but still work, ~~and~~ dep on z
- here, too $Z(\tilde{\mathfrak{S}}_\lambda)$ gives Whit fcns. It's a bit messier:

$$Z(\tilde{\mathfrak{S}}_\lambda ; \underline{c}) = z^{\text{factor}(c,N)} W_z^\gamma(\varpi^\lambda).$$

- Also mention mx business.