Iwahori-Hecke Algebras in Multiple Contexts

1) Hecke algebras for a reductive group
2) Presentation of spherical/finite/affine Hecke algebras
3) Quantum Schur-Weyl duality

1) Reductive Groups

	Favorite example
G: reductive gp. / F: nonarch local field	$G = GL_n(\mathbb{Q}_p)$
$\mathcal{O}$: ring of integers of F	$\mathcal{O} = \mathbb{Z}_p$
$\mathfrak{p}$: maximal ideal of $\mathcal{O}$	$\mathfrak{p} = \langle p \rangle$
B = Borel subgp.	$B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$
K^0 = maximal compact subgp.	$K^0 = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}$
J = Iwahori subgp.	$J = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ & \ddots & \vdots \\ \mathfrak{p} & & \mathcal{O} \end{pmatrix}$

Let K be a compact open subgp. of G. The Hecke algebra of G relative to K is the set of smooth, compactly supported K-biinvariant functions on G:

$$\mathcal{H}_K := \{\phi : G \to \mathbb{C}, \text{ smooth, cpt. supp} \mid \phi(kgk') = \phi(g) \ \forall k, k' \in K, g \in G\},$$

w/ mult. defined by convolution.

1) Reductive gps. are hard
2) Hecke algebras are relatively simple: often finite(-ish) dim'l (see next section)
3) Borel-Matsumoto: $\exists$ corresp. btwn irreps $\mathcal{H}_K$ and "smooth, admissible" irreps of G w/ K-fixed vector v ($k \cdot v = v \ \forall k \in K$).
4) So Hecke alg's. are a tool to understand the rep'n theory of reductive gps.

But what do Hecke algebras actually look like?

2) Presentations (Iwahori)

For this section, $G = GL_n$, but can be done for any Cartan type.

$$\mathcal{H}_{K^\circ} \cong X_*(T) \cong \mathbb{Z}^n \quad \text{(spherical Hecke alg.}$$

($*$ ← cochar lattice)

$$\mathcal{H}_B = \left\langle T_i,\ i = 1, \dots, n-1 \;\middle|\; \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ T_i T_j = T_j T_i,\ i \neq j \pm 1 \\ T_i^2 = (q-1) T_i + q \end{array} \right\rangle$$

(finite Hecke alg.)

$$\mathcal{H}_J = \left\langle T_i,\ i = 0, \dots, n-1 \;\middle|\; \begin{array}{l} \text{Same rel'ns as for} \\ \mathcal{H}_B \text{, but indices mod } n \end{array} \right\rangle$$

(affine Hecke algebra)

Remarks

1) <u>Not</u> guaranteed a simple presentation of $\mathcal{H}_K$ for other sub gps. K, but ...

2) $\mathcal{H}_{K^\circ}$ is commutative!

3) $\mathcal{H}_B$ is finite dim'l, is a deformation of the group alg. of S_n (finite Coxeter gps. in general):

$$\text{If } q \mapsto 1$$
$$\mathcal{H}_B \mapsto \mathbb{C}[S_n]$$

So repn theory of finite Hecke alg. relates to repn. theory of S_n.

4) Exact sequences:

$$1 \to \mathcal{O}K^\circ \to K^\circ \to B(\mathbb{F}_q) \to 1$$
$$\wr$$
$$0 \to \mathcal{H}_{K^\circ} \to \mathcal{H}_J \to \mathcal{H}_B \to 0$$

So to understand $\mathcal{H}_J$, want to understand $\mathcal{H}_{K^\circ}, \mathcal{H}_B$.

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3) Quantum Schur-Weyl Duality

First, classical S-W duality:

Let $V = \mathbb{C}^n$ be the std. repn of $G = GL_n(\mathbb{C})$.

Now take $V^{\otimes k}$ for $k \leq n$, and let G act diagonally:

$$g \cdot (v_1 \otimes \cdots \otimes v_k) = g \cdot v_1 \otimes \cdots \otimes g \cdot v_k .$$

Let S_k act on $V^{\otimes k}$ by permuting the factors:

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

These actions commute, and in fact are <u>mutual centralizers</u>.

Schur-Weyl duality: As a (GL_n, S_k)-bimod., $V^{\otimes k}$ decomposes as

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} L^\lambda \otimes S^\lambda ,$$

where the L^λ are (distinct) highest wt. repns, and the S^λ are (distinct) Specht modules.

Now, let V be std. repn. of the quantum gp. $U = U_q(\mathfrak{gl}_n)$, $q \neq$ root of unity, and let U act on $V^{\otimes k}$ by the coproduct map.

Since U not "cocomm", we can't just permute the factors as before. Instead, we use the Yang-Baxter eqn. to define isomorphisms

$$R_i : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_k \xrightarrow{\sim} V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_k .$$

Thm (Jimbo '86): The alg. gen'd by the R_i is isom. $\mathcal{H}_B$ (for GL_k), and the U and $\mathcal{H}_B$ actions are mutual centralizers.

We have the decomp.

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} L_q^\lambda \otimes S_q^\lambda ,$$

where the L_q^λ, S_q^λ are irred, and deformations of the L^λ, S^λ.

Remarks

1) Jimbo's results helped kick-start huge breakthroughs. One notable example: Jones' Fields Medal work on knot invariants.

2) This section only holds for GL_n, not a reductive group of any other type.

3) Not surprising that $U_q(\mathfrak{gl}_n)$ is in S-W duality w/ a deformation of $\mathbb{C}[S_n]$, but it is remarkable that this deformation turns out to be the Hecke algebra.

4) I am not aware of any "natural" (functorial) for remark 3, and in light of remark 2), might be hard to have a general result, Would be very interesting if such a result exists!