Representation Stability, Étale Cohomology and Combinatorics of Configuration Spaces over Finite Fields

Following Church-Ellenberg-Farb, *Representation stability in cohomology and asymptotics for families of varieties over finite fields*

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Configuration spaces: Classically

Definition

- $\mathrm{PConf}_n(F) = \{(x_1, \ldots, x_n) \in F^n \ : \ x_i \neq x_j \text{ when } i \neq j\}$
- $\mathrm{Conf}_n(F) = \mathrm{PConf}(F)/S^n$

Configuration Spaces: As Schemes

Let...

- D_n be the space of monic degree-n polynomials in T
- $\pi : \mathbb{A}^n \to D_n$ by $(x_1, \dots, x_n) \mapsto (T - x_1) \dots (T - x_n)$

S^n acts $\mathbb{A}^n$ with $\pi(\sigma x) = \pi(x)$ for $\sigma \in S_n$. $D_n = \mathbb{A}^n / S_n$

Definition

- $\mathrm{Conf}_n := D_n \setminus V(\Delta)$ where Δ is the discriminant
- $\mathrm{PConf}_n := \mathbb{A}^n \setminus \bigcup_{i<j} V(x_j - x_i)$

Note $\pi : \mathrm{PConf}_n \to \mathrm{Conf}_n$

Configuration Spaces: A Warning

$\mathrm{Conf}_n \cong \mathrm{PConf}_n/S_n$ **scheme-theoretically.**

Our Plan

- Recall $\mathrm{PConf}_n(\mathbb{C})$ is an analytic manifold.
- $H^i(\mathrm{PConf}_n(\mathbb{C}))$ is an S_n-representation...
- ...with maps between induced by $\mathrm{PConf}_{n+1} \to \mathrm{PConf}_n$
- We use the theory of FI-modules study $\chi_{H^i(\mathrm{PConf}_n(\mathbb{C}))}$ as $n \to \infty$ ...
- ...and use Grothendieck-Lefschetz relate it to combinatorics on $\mathrm{Conf}_n(\mathbb{F}_q)$????

Ingredient: Étale Homotopy Theory

Goal: relate $H^*(\mathrm{PConf}_n(\mathbb{C}))$ to $H^*(\mathrm{Conf}_n(\mathbb{F}_q))$

Problem: Zariski topology and singular cohomology are not friends

Solution: Étale Cohomology

Following [Mil13], [Gro13].

A Brief Introduction to Sites

An (Extremely) Brief Introduction to Sites

Definition

A *Grothendieck Topology* on a category $\mathcal{C}$ consists of… for each $U \in \mathcal{C}$ a distinguished set of *coverings* $(U_i \to U)_{i \in I}$ such that

- various axioms are fulfilled
- which imitate the properties of $\mathrm{Op}(X)$

Such a category $\mathcal{C}$ equipped with a Grothendieck Topology is called a *site.*

The Étale Site

Definition

A morphism of varieties $f : X \to Y$ is *Étale* if it is smooth and unramified.
When X and Y are smooth, this is equivalent to inducing an isomorphism $T_x X \to T_y Y$ for each closed point $y \in Y$ and $x \in f^{-1}(y)$.

Definition

Let $\mathrm{Et}(X)$ be the category of étale maps with target X.
Declare our coverings to be surjective families $(U_i \to U)_{i \in I}$

Étale Cohomology

Definition

- An *étale presheaf $\mathcal{F}$ on X* is a functor $\mathrm{Et}(X)^{\mathrm{op}} \to \mathrm{Ab}$.
- An étale sheaf is an étale presheaf which satisfies site-theoretic analogues of the sheaf axioms.
- Denote the category of étale sheaves on X by $\mathrm{Sh}^{\text{ét}}(X)$
- $H^i_{\text{ét}}(X; \mathcal{F})$ is defined as $R^i(\Gamma)(\mathcal{F})$ for an étale sheaf $\mathcal{F}$

Étale Cohomology

Let ℓ be a prime and $\underline{\mathbb{Z}/\ell^k}$ the constant sheaf with value $\mathbb{Z}/\ell^k$.

Definition (ℓ-adic Cohomology)

Define $H^i_{\text{ét}}(X; \mathbb{Z}_\ell) := \varprojlim H^i_{\text{ét}}(X; \underline{\mathbb{Z}/\ell^k})$

$H^i_{\text{ét}}(X; \mathbb{Q}_\ell) := H^i_{\text{ét}}(X; \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

A Technical Note

Notation

- *Henceforth, when taking étale cohomology, X will be a variety defined over $\mathbb{F}_q$.*
- *$H^i_{\text{ét}}(X;\mathbb{Q}_\ell)$ will be shorthand for $H^i_{\text{ét}}(X_{/\bar{\mathbb{F}}_q};\mathbb{Q}_\ell)$, with $X_{/\bar{\mathbb{F}}_q}$ denoting the base change of X to $\bar{\mathbb{F}}_q$.*
- $$X_{/\bar{\mathbb{F}}_q} = X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,\bar{\mathbb{F}}_q.$$

The Comparison Map

Let X be a nonsingular variety defined over $\mathbb{Z}$ and G a finite Abelian group.

Fact (nontrivial)**:** There exists a map $H^i_{\text{ét}}(X_{/\bar{\mathbb{F}}_q}; \underline{G}) \to H^i(X(\mathbb{C}); G)$

Theorem (Artin)

Under the conditions above, $H^i_{\text{ét}}(X_{/\bar{\mathbb{F}}_q}; \underline{G}) \to H^i(X(\mathbb{C}); G)$ *is an isomorphism.*

Taking limits and tensoring with $\mathbb{Q}_\ell$*,* $H^i_{\text{ét}}(X_{/\bar{\mathbb{F}}_q}; \mathbb{Q}_\ell) \to H^i(X(\mathbb{C}); \mathbb{Q}_\ell)$ *is an isomorphism as well.*

Grothendieck-Lefschetz

Let Y be a compact topological space and $f : Y \to Y$.

Theorem (Lefschetz Fixed-Point)

$\#\mathrm{Fix}(f : Y \to Y) = \sum_{i \geq 0} (-1)^i \operatorname{tr}(f^* : H^i(Y, \mathbb{Q}))$

Grothendieck: apply this to $\mathrm{Frob}_q : X_{/\bar{\mathbb{F}}_q} \to X_{/\bar{\mathbb{F}}_q}$ via étale cohomology.

Recall: $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is generated by Frob_q. $\implies$ $|X(\mathbb{F}_q)| = \#\mathrm{Fix}(\mathrm{Frob}_q)$

Grothendieck-Lefschetz

Theorem (Grothendieck-Lefschetz; [Gro77])

For any smooth projective variety X over $\mathbb{F}_q$,

$$|X(\mathbb{F}_q)| = \#\,\mathrm{Fix}(\mathrm{Frob}_q) = \sum_{i\geq 0}(-1)^i\,\mathrm{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X;\mathbb{Q}_\ell))$$

If X is smooth but not projective, Poincaré duality implies

$$|X(\mathbb{F}_q)| = q^{\dim X}\sum_{i\geq 0}(-1)^i\,\mathrm{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X;\mathbb{Q}_\ell)^*)$$

Grothendieck-Lefschetz: An Example

Theorem (Grothendieck-Lefschetz (Non-Projective))

$$|X(\mathbb{F}_q)| = q^{\dim X} \sum_{i \geq 0} (-1)^i \operatorname{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X; \mathbb{Q}_\ell)^*)$$

Example ($|\mathrm{Conf}_n(\mathbb{F}_q)|$)

Fact: Frob_q acts on $H^i_{\text{ét}}(\mathrm{Conf}_n; \mathbb{Q}_\ell)$ by multiplication by q^i and hence on $H^i_{\text{ét}}(\mathrm{Conf}_n; \mathbb{Q}_\ell)^*$ by q^{-i}

Arnold: $H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C}) = \mathbb{C}$ when $i = 0, 1$ and 0 otherwise

$$\implies \operatorname{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(\mathrm{Conf}_n; \mathbb{Q}_\ell)^*) = \begin{cases} 1 & i = 0 \\ q^{-1} & i = 1 \end{cases}$$

$$\implies |\mathrm{Conf}_n(\mathbb{F}_q)| = q^n(1 - q^{-1}) = q^n - q^{n-1}$$

Étale Fundamental Group

Classical: For $x \in X$, let Fib_x be the functor $\mathrm{Cov}(X) \to \mathrm{Set}$ with $\mathrm{Fib}_x(Y)$ defined for $\pi : Y \to X$ as $\pi^{-1}(x)$.

Fact: $\pi_1(X)$ acts transitively and faithfully on Fib_x by *mondronomy action*

Definition

$$\pi_1^{\mathrm{et}}(X, x) := \mathrm{Aut}_{\mathrm{Set}^{\mathrm{Et}(X)}}(\mathrm{Fib}_x^{\text{ét}}).$$

Local Systems & ℓ-adic Sheaves

Recall: A local system (classically) is a locally constant sheaf of Abelian groups.

For $x \in X$, L is an $\mathrm{Aut}(A)$-local system if $L_x \cong A$.

Fact: There is an equivalence of categories between $\mathrm{Aut}(A)$-local systems and representations $\pi_1(X) \to \mathrm{Aut}(A)$

Definition

For G a topological group, an étale G-local system is a representation $\pi_1^{\mathrm{et}}(X, x) \to G$.
An ℓ-adic sheaf is an étale $\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$-local system

Local Systems & ℓ-adic Sheaves

The Analogy:

Classically: $\mathrm{GL}_n(\mathbb{F})$-local systems $\longleftrightarrow$ $\mathbb{F}$-vector bundles with flat connection

Étale: ℓ-adic sheaves are "like" vector bundles with flat connection.

Grothendieck-Lefschetz with Twisted Coefficients

Theorem (G-L With Twisted Coefficients)

For an ℓ-adic sheaf $\mathcal{F}$ on projective X,

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q \mid \mathcal{F}_x) = \sum_i (-1)^i \,\mathrm{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X; \mathcal{F})).$$

In the non-projective case:

$$\sum_{x \in X(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q \mid \mathcal{F}_x) = q^{\dim X} \sum_i (-1)^i \,\mathrm{tr}(\mathrm{Frob}_q : H^i_{\text{ét}}(X; \mathcal{F})^*)$$

The FI Category

Definition

Let FI denote the category with objects **F**inite sets and morphisms **I**njections.

FI is equivalent to its skeletal subcategory with objects $\mathbf{n} := \{1, \ldots, n\}$

FI-Objects

Definition

- A FI-module over a commutative ring R is a functor $V : \mathsf{FI} \to \mathrm{Mod}_R$. We denote $V(\mathbf{n}) =: V_n$.
- More generally, a FI-[object] (resp. $\mathsf{FI}^{\mathrm{op}}$-[object]) W is a functor $W : \mathsf{FI} \to$ [objects] (resp. $\mathsf{FI}^{\mathrm{op}}$)

Remark

$\mathrm{End}_{\mathrm{FI}}(\mathbf{n}) = S_n$. Hence, a FI-module defines a sequence of S_n representations in a "coherent" manner.

FI-Examples

Example

$V : \mathsf{FI} \to \mathrm{Vect}_{\mathbb{R}}$ with $V_n = \langle x_1, \ldots, x_n \rangle$ and $V(\sigma) : x_k \mapsto x_{\sigma(k)}$ is a FI-Module.

Example

$C : \mathsf{FI}^{\mathrm{op}} \to \mathrm{Top}$ with $C_n = \mathrm{PConf}_n(\mathbb{C})$ and for $\sigma : \mathbf{m} \hookrightarrow \mathbf{n}$ $C(\sigma) : (z_1, \ldots, z_n) \mapsto (z_{i(1)}, \ldots, z_{i(m)})$ is a $\mathsf{FI}^{\mathrm{op}}$-space.

It follows that $H^i \circ C$ is a FI-module!

Finite Generation of FI-modules

Definition

A FI-module V is finitely generated if there are finitely many elements $x_1, \ldots, x_n \in \bigcup_{i \geq 0} V_n$ such that each V_n is generated by FI-images of the x_i.

Remark

Each of our examples are finitely-generated!

Character Polynomials

Recall that the conjugacy class of $\sigma \in S_n$ is determined by $(c_1(\sigma), c_2(\sigma), \dots)$ where $c_i(\sigma) := \#i$-cycles in σ.

Definition

- A character polynomial P is an element of the ring $\mathbb{Q}[X_1, X_2, \dots]$ graded by $|x_i| = i$.
- We think of P as giving a sequence of S_n-characters!

Character Polynomials

A sequence of S_n-characters $\{\chi_n\}$ is *given by the character polynomial* P if for $\sigma \in S_n$, $\chi_n(\sigma)$ coincides with the class function $P_n : S_n \to \mathbb{Q}$ defined by
$P_n(\sigma) = P(c_1(\sigma), c_2(\sigma), \dots)$

Character Polynomials

A sequence of S_n-characters $\{\chi_n\}$ is **eventually** *given by the character polynomial* P if there exists N such that for $n > N$ and $\sigma \in S_n$, $\chi_n(\sigma)$ coincides with the class function $P_n : S_n \to \mathbb{Q}$ defined by $P_n(\sigma) = P(c_1(\sigma), c_2(\sigma), \dots)$

Two Theorems on Character Polynomials

Theorem ([CEF14, 3.9])

Given two character polynomials $P, Q \in \mathbb{Q}[X_1, \dots]$, $\langle P_n, Q_n \rangle_{S_n}$ is independent of n when $n \geq \deg P + \deg Q$.

We denote $\langle P, Q \rangle := \lim_{n\to\infty} \langle P_n, Q_n \rangle_{S_n}$.

Theorem ([CEF15, 3.3.4])

Let V be a finitely generated FI-module over a field of characteristic zero and let $\chi_V = \{\chi_n\}$ be its sequence of characters. χ_V is eventually given by a unique character polynomial P_V.

STATISTICS ON $\mathrm{Conf}_n(\mathbb{F}^q)$

- Our focus: statistics depending on the length of irreducible factors in elements of $\mathrm{Conf}_n(\mathbb{F}_q)$.
- Let $\chi : S^n \to \mathbb{Q}$ be a class function and $f \in \mathrm{Conf}_n(\mathbb{F}^q)$. Let $R(f) = \{x \in \overline{\mathbb{F}}_q : f(x) = 0\}$.
- Frob_q induces a permutation σ_f on $R(f)$ (defined up to conjugacy).
- $\chi(f) := \chi(\sigma_f)$.

Relating Statistics and Homology I

Theorem ([CEF14, 3.7])

Let χ be any class function $S_n \to \mathbb{Q}$. Then,

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \chi(f) = \sum_i (-1)^i q^{n-i} \langle \chi, H^i(\mathrm{PConf}_n(\mathbb{C})) \rangle$$

Relating Statistics and Homology II

Sketch.

- Restrict focus to χ irreducible
- $\mathrm{PConf}_n \to \mathrm{Conf}_n$ is a Galois cover with deck group S_n.
- (Non-trivial) yields [f.d. representations of S_n] $\cong$ [f.d. local systems on Conf_n trivial on PConf_n]
- Let V correspond to χ and $\mathcal{V}$ be the corresponding local system.
- G-L:

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Frob}_q : \mathcal{V}_f) = \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \chi(f)$$

$$= q^n \sum_j (-1)^j \, \mathrm{tr}(\mathrm{Frob}_q : H^j_{\text{ét}}(\mathrm{Conf}_n; \mathcal{V})^*)$$

□

Relating Statistics and Homology III

Sketch (Cont.)

- Know Frob_q acts on $H^j_{\text{ét}}(\mathrm{Conf}_n;\mathcal{V})^*$ by q^{-i}; just need $\dim_{\mathbb{Q}_\ell} H^j_{\text{ét}}(\mathrm{Conf}_n;\mathcal{V})^*$.
- Pull back $\mathcal{V}$ to $\tilde{\mathcal{V}}$ on PConf_n
- $$\begin{aligned} H^j_{\text{ét}}(\mathrm{Conf}_n;\mathcal{V})^* &\cong (H^j_{\text{ét}}(\mathrm{PConf}_n;\tilde{\mathcal{V}})^*)^{S_n} \\ &\cong (H^j_{\text{ét}}(\mathrm{PConf}_n;\mathbb{Q}_\ell)^* \otimes V)^{S_n} \\ &\cong H^j_{\text{ét}}(\mathrm{PConf}_n;\mathbb{Q}_\ell)^* \otimes_{\mathbb{Q}_\ell[S_n]} V \end{aligned}$$
- Rep. theory: $\dim(H^j_{\text{ét}}(\mathrm{PConf}_n;\mathbb{Q}_\ell)^* \otimes_{\mathbb{Q}_\ell[S_n]} V) = \langle \chi, H^j_{\text{ét}}(\mathrm{PConf}_n;\mathbb{Q}_\ell)\rangle$
- **Fact:** Artin's comparison map is an isomorphism of S_n-representations.

□

Stabilization

Let P be a character polynomial and denote by $\langle P, H^i(\mathrm{PConf})\rangle = \lim_{n\to\infty}\langle P_n, H^i_{\text{ét}}(\mathrm{PConf}_n; \mathbb{Q}_\ell)\rangle$.

Theorem ([CEF14, 3.13])

The following limit exists:

$$\lim_{n\to\infty} q^{-n} \sum_{f\in \mathrm{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty}(-1)^i \frac{\langle P, H^i(\mathrm{PConf})\rangle}{q^i}$$

The Braid Group

- B_n: the braid group on n strands. PB_n: the pure braid group on n strands
- $1 \to PB_n \to B_n \to S_n \to 1$ is exact
- **Recall:** $\mathrm{PConf}_n(\mathbb{C})$ is a $K(\pi, 1)$ with $\pi = PB_n$. $\mathrm{Conf}_n(\mathbb{C})$ is as well for $\pi = B_n$
- Thus, $H^i(\mathrm{PConf}_n(\mathbb{C})) = H^i(PB_n)$.

The Braid Group

Theorem ([CEF14, 4.1 & 4.3])

Let χ be a S_n-character

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \chi(f) = \sum_i (-1)^i q^{n-i} \langle \chi, H^i(PB_n) \rangle$$

Further, the inner product $\langle P, H^i(PB_n) \rangle$ is independent of n for $n \geq 2i + \deg P$ and

$$\lim_{n \to \infty} q^{-n} \sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} P(f) = \sum_{i=0}^{\infty} (-1)^i \frac{\langle P, H^i(\mathrm{PConf}) \rangle}{q^i}$$

Example: Linear factors

Theorem ([CEF14, 4.4])

The expected number of linear factors for a monic, squarefree degree-n polynomial in $\mathbb{F}_q[t]$ approaches $\sum_{i=0}^{\infty} \frac{(-1)^i}{q^i}$

Sketch.

- Recall: $X_1(f) = c_1(\sigma_f)$ is the # of linear factors of f.
- **Fact:** when $i > 0$, $\langle X_1, H^i(P_n)\rangle$ is 0 when $n < i+1$, 1 when $n = i+1$ and 2 when $n > i+1$.
- Theorem $\implies$

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} X_1(f) = q^n - \frac{2}{q^{n-1}} + \frac{2}{q^{n-2}} + \cdots \pm 2q^2 \mp q$$

- Divide through by $|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n - q^{n-1}$, take a limit.

Thank you for listening!

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