Let $g(x)$ be defined near x_0, say on $|x-x_0| \leq h$. Assume that

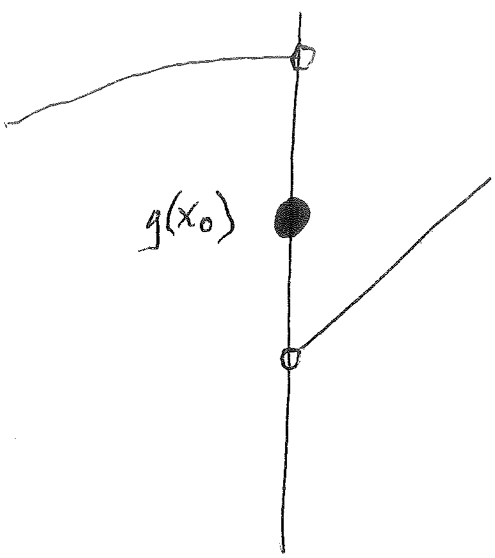
$$g(x_0+0) \quad \text{and} \quad g(x_0-0)$$

make sense.

Def. of One-Sided Derivatives:

$$g_+'(x_0) = \lim_{s\to 0^+} \frac{g(x_0+s)-g(x_0+0)}{s};$$

$$g_-'(x_0) = \lim_{s\to 0^-} \frac{g(x_0+s)-g(x_0-0)}{s}.$$



$$g_-'(x_0) \approx .01$$
$$g_+'(x_0) \approx 1.0$$

Suppose that $g(x)$ is piecewise C^1 and that h is so tiny that $g'(x_0+s)$ exists and is continuous for $0 < |s| \leq h$.

We can apply L'Hôpital! (Baby Calc)

$$g_+'(x_0) = \lim_{s \to 0^+} \frac{g(x_0+s) - g(x_0+0)}{s}$$

$$= \lim_{s \to 0^+} \frac{g'(x_0+s)}{1}$$

$$= g'(x_0+0) .$$

Yes!

Similarly

$$g_-'(x_0) = g'(x_0-0) .$$

Dirichlet Kernel : $-\infty < t < \infty$

$$D_N(t) \equiv \frac{1}{2} + \cos(t) + \cdots + \cos(Nt) \,.$$

Easy to prove :

(a) $D_N(t)$ is 2π - periodic

(b) $D_N(t)$ is even

(c) $D_N(0) = N + \frac{1}{2}$

(d) $D_N(t) = \dfrac{\sin(N+\frac{1}{2})t}{2\sin(\frac{t}{2})}$ $t \neq 2k\pi$

(e) $\int_0^\pi D_N(t)\,dt = \frac{\pi}{2}$.

book p. 32

a, b, c, e trivial!

HW??

(d) can be done by induction, <u>or</u> <u>else</u> <u>complex</u> <u>numbers</u>!

$i = \sqrt{-1}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \cos\theta + i\sin\theta$$

{plug in Taylor series!}

$$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$$ <u>law of exponents</u>

{multiply out; use trig}

$$\therefore \quad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \text{any } \theta$$

$$\Downarrow$$

$$D_N(\theta) = \frac{1}{2} + \sum_{k=1}^{N} \frac{1}{2}\left(e^{ik\theta} + e^{-ik\theta}\right)$$

$$= \frac{1}{2} \sum_{k=-N}^{N} e^{ik\theta}$$

$$= \frac{1}{2} \sum_{k=-N}^{N} \left(e^{i\theta}\right)^k$$

Geometric Progression $(r \neq 1)$

$$1 + r + r^2 + \cdots + r^{M-1} = \frac{1-r^M}{1-r}$$

$r = e^{i\theta}$. Have:

$$\frac{1}{2}\left[r^{-N} + r^{-N+1} + \cdots + r^N\right]$$

$$= \frac{1}{2} r^{-N} \left(1 + r + r^2 + \cdots + r^{2N}\right)$$

$$= \frac{1}{2} r^{-N} \frac{1 - r^{2N+1}}{1 - r} \qquad (r \neq 1)$$

$$= \frac{1}{2} \frac{r^{-N} - r^{N+1}}{1 - r}$$

$$\left\{ r^k = (e^{i\theta})^k = e^{ik\theta} \right\}$$

$$= \frac{1}{2} \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \qquad \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}$$

$$= \frac{1}{2} \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}$$

$$\left\{ e^{-i\phi} - e^{i\phi} = -2i\sin\phi \, , \quad \text{any } \phi \right\}$$

$$= \frac{1}{2}\ \frac{-2i\sin(N+\frac{1}{2})\theta}{-2i\sin(\frac{\theta}{2})}$$

$$= \frac{1}{2}\ \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$$

$r \neq 1$ means $e^{i\theta} = \cos\theta + i\sin\theta \neq 1$

i.e. $\theta \neq \underline{2k\pi}$

So,

$$D_N(\theta) = \frac{\sin(N+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}, \quad \theta \neq 2k\pi.$$

(d) is OK!

QED ∎

Let $h(x)$ be given on $[a, a+2\pi]$, say. Piecewise continuous.

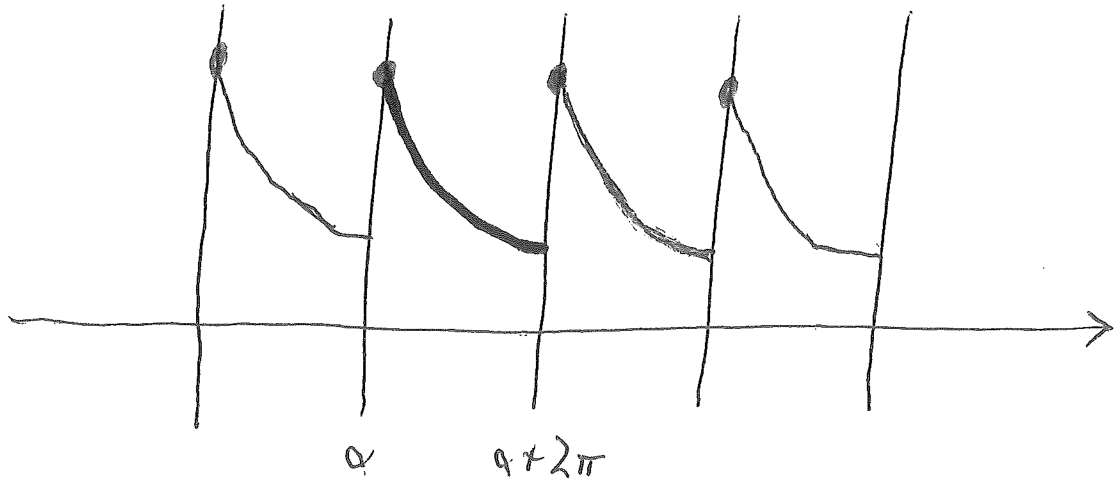
We seek the 2π-periodic extension of $h(x)$. $\boxed{\text{Needed for Fourier's Thm.}}$

Need a 2π-periodic function that "extends" $h(x)$. * $\boxed{\text{ANALYTIC DEF.}}$

Don't worry about endpoint values of h. We simply adjust them so that $h(a) = h(a+2\pi)$. E.g., take both to be 0.

$\boxed{\text{* This is trivial graphically!!}}$

*



Note

$$[\alpha, \alpha+2\pi) = \{\alpha \leq x < \alpha+2\pi\}$$

$$\mathbb{R} = \bigcup_{m=-\infty}^{\infty} [\alpha + 2\pi m, \alpha + 2\pi(m+1))$$

disjoint intervals!

Consider (ANY) $x_0 \in \mathbb{R}$. Get unique k so that

$$\alpha + 2\pi k \leq x_0 < \alpha + 2\pi k + 2\pi$$

i.e. $\alpha \leq x_0 - 2\pi k < \alpha + 2\pi$.

We just define

$$H(x_0) \equiv h(x_0 - 2\pi k).$$

This works!!

$\alpha \leq x_0 < \alpha + 2\pi$ leads to $k = 0$. OK

OK

So, $H(x) = h(x)$ for $a \leq x < a + 2\pi$.

Also, for general x_0, note $x_0 + 2\pi$ "gets" #$k+1$. So,

$$H(x_0 + 2\pi) = h\left[x_0 + 2\pi - 2\pi(k+1)\right]$$

$$= h(x_0 - 2\pi k)$$

$$= H(x_0).$$

So, $H(x)$ is 2π-periodic. OK

I just explained how to form the $\underline{2\pi}$-periodic extension of $h(x)$ given on $[\alpha, \alpha + \underline{2\pi}]$. $\underline{H}(x)$ *

Similarly $[\alpha, \alpha + 2L]$ and $2L$-periodic extension.

Back to $L = \pi$

$[\alpha + 2\pi k, \alpha + 2\pi(k+1))$

$\Rightarrow \underline{H}(x) \equiv h(x - 2\pi k)$ <u>def.</u>

h piecewise continuous $\Rightarrow$ H likewise

h <u>piecewise C^1</u> $\Rightarrow$ <u>H likewise</u>

etc.

*

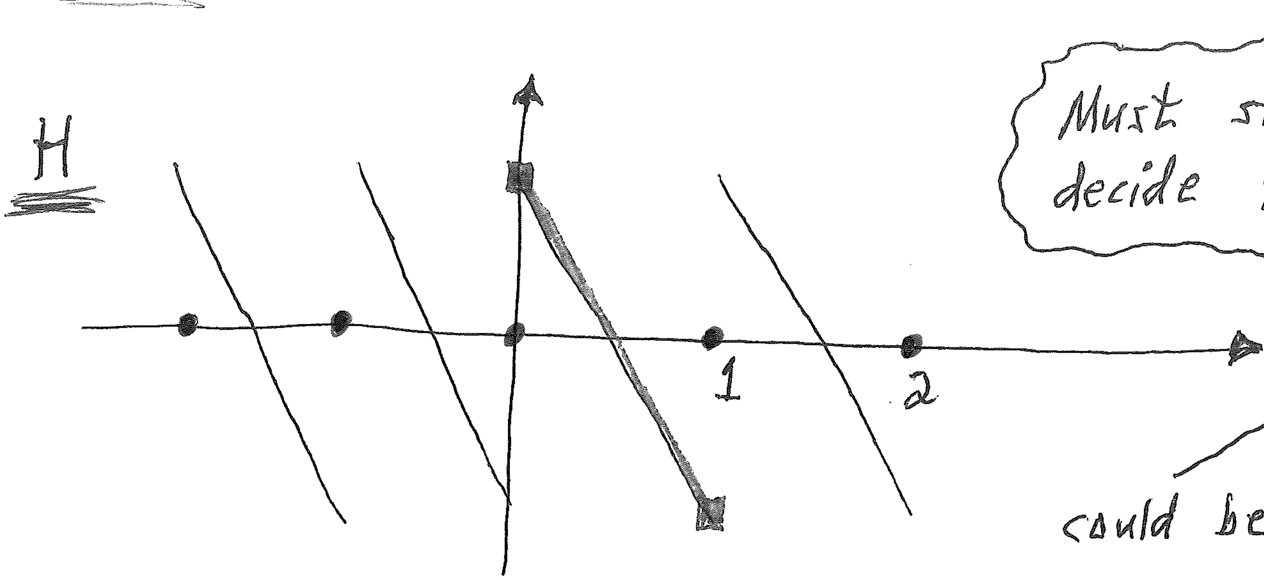
[Always prepared to "fudge" value at points of form $\alpha + \underline{2\pi m}$]

(NOTE)

There is no question: at **each** $x_0 \in \mathbb{R}$, the values of 1-sided limits $H(x_0+0)$, $H(x_0-0)$ are **unambiguous** insofar as the given $h(x)$ is **piecewise continuous**.

Example of $H(x)$. $2L = 1$ $[0,1]$

$h(x) = 1 - 2x$, $[0,1]$, $r = 0$



Must still decide $H(m)$

could be 0! or 13!!

SAW —

$$1-2x \sim \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin(2\pi n x)$$

$FS(h)$

$$\frac{\pi n}{1/2}$$

Notice: what is value of RHS at $x = M =$ any integer? $\underline{\underline{0}}$

Notice on graph of $H(x)$:

$$H(M+0) = 1$$
$$H(M-0) = -1$$

and
$$\frac{H(M+0) + H(M-0)}{2} = \underline{\underline{0}}.$$

curious

Fourier's Theorem (early form)

Let $h(x)$ be piecewise C^1 on $[\alpha, \alpha + 2L]$. Form $2L$-periodic extension of $h(x)$; call it $H(x)$.

Form FS of $h(x)$ — FS(h)

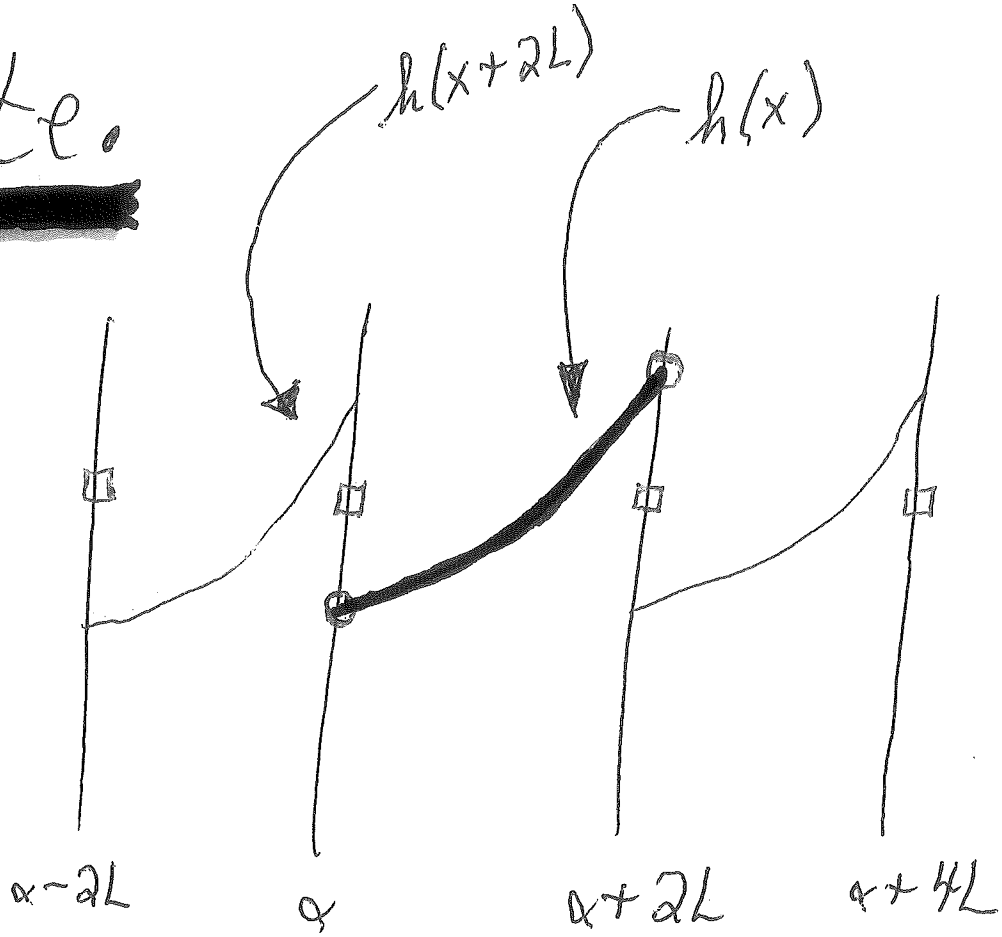
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi n x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Then: the FS(h) converges at each point $x_0 \in \mathbb{R}$ to

$$\frac{H(x_0+0) + H(x_0-0)}{2}.$$

Standard / easy / common form.

Note.



cf. □

At $x_0 = \alpha$, get

$$\frac{h(\alpha+0) + h[(\alpha+2L)-0]}{2}.$$

x_1 = JUMP in $(\alpha, \alpha+2L)$?? Get:

ok $$\frac{h(x_1+0) + h(x_1-0)}{2}.$$

Re: Fourier's Thm

$h(t)$ on $[\alpha, \alpha + 2\pi]$, $L = \pi$, say.

$$S_N(x_0) = \frac{1}{2} A_0 + \sum_{n=1}^{N} \left(A_n \cos n x_0 + B_n \sin n x_0 \right)$$

$$A_n = \frac{1}{\pi} \int_I h(t) \cos(nt)\, dt$$

$$B_n = \frac{1}{\pi} \int_I h(t) \sin(nt)\, dt$$

$I \equiv [\alpha, \alpha + 2\pi]$

So,

$$S_N(x_0) = \frac{1}{\pi} \int_I h(t) \left\{ \frac{1}{2} + \sum_{n=1}^{N} \cos nt \cos n x_0 + \sum_{n=1}^{N} \sin nt \sin n x_0 \right\} dt$$

book p. 24 prob 9

$$S_N(x_0) = \frac{1}{\pi} \int_I h(t) \left\{ \frac{1}{2} + \sum_{1}^{N} \cos n(t - x_0) \right\} dt$$

from p. 3

$$= \frac{1}{\pi} \int_I h(t)\, D_N(t - x_0)\, dt$$

KEY!

$$= \frac{1}{\pi} \int_I \underline{H(t)}\, D_N(t - x_0)\, dt$$

$$\left\{ \begin{array}{l} \underline{\text{take } \; t = x_0 + v, \quad v = t - x_0} \\ \text{so,} \quad a - x_0 \leq v \leq a + 2\pi - x_0 \end{array} \right\}$$

$$= \frac{1}{\pi} \int_{a - x_0}^{a + 2\pi - x_0} H(x_0 + v)\, D_N(v)\, dv$$

KEY #2!

<u>2π</u>-periodic

$$\boxed{\text{CAN USE } \underline{\text{ANY}} \; v\text{-interval OF LENGTH } 2\pi}$$

So, we have:

$$S_N(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi} H(x_0+v)\, D_N(v)\, dv \, .$$

slick.

{ so far so good! }

Need: limit as $N \to \infty$ to be

$$\frac{1}{2} H(x_0+0) + \frac{1}{2} H(x_0-0) \, .$$

Try to express this (sum) as an integral.

$$\frac{1}{\pi}\int_0^{\pi} D_N(v)\, dv = \frac{1}{2}$$

$$\frac{1}{\pi}\int_{-\pi}^{0} D_N(v)\, dv = \frac{1}{2}$$

Aha!

$$\frac{1}{\pi}\int_0^{\pi} \underline{H(x_0+0)}\, D_N(v)\,dv = \frac{H(x_0+0)}{2}$$

$$\frac{1}{\pi}\int_{-\pi}^{0} \underline{H(x_0-0)}\, D_N(v)\,dv = \frac{H(x_0-0)}{2}$$

So, we have: use $\underline{2}$ integrals

$$S_N(x_0) - \frac{1}{2}H(x_0+0) - \frac{1}{2}H(x_0-0)$$

$$= \frac{1}{\pi}\int_0^{\underline{\pi}} \left[H(x_0+v) - \underline{H(x_0+0)}\right] D_N(v)\,dv$$

$$+ \frac{1}{\pi}\int_{\underline{-\pi}}^{0} \left[H(x_0+v) - \underline{H(x_0-0)}\right] D_N(v)\,dv.$$

Very similar pieces!

We want to show each piece $\to 0$ as $N \to \infty$.

Note: Will use R-L Lemma!!

Piece 1 is:

$$\frac{1}{\pi}\int_0^{\pi}\left[H(x_0+v)-H(x_0+0)\right]\frac{\sin(N+\frac{1}{2})v}{2\sin(\frac{v}{2})}\,dv$$

would prefer $2(\frac{v}{2}) = v$

i.e.

$$\frac{1}{\pi}\int_0^{\pi}\frac{H(x_0+v)-H(x_0+0)}{v}\;\frac{v}{2\sin(\frac{v}{2})}\;\sin(N+\tfrac{1}{2})v\,dv$$

$\ast$ $\dfrac{v/2}{\sin(\frac{v}{2})}$

[integrand at $v \simeq 0$ irrelevant]

$v > 0$

Pause for some elem calc!!

$$\left\{ \begin{array}{ll} 1, & y = 0 \\ \\ \dfrac{\sin(y)}{y}, & 0 < y \leq \underline{\dfrac{\pi}{2}} \end{array} \right\}$$

N.B. $\int_0^1 \cos(yu)\,du$

is a continuous <u>NONZERO</u> function. In fact, it is $\underline{C^\infty}$. Indeed,

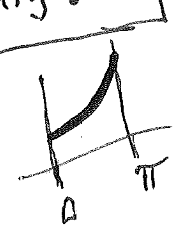
$$\frac{\sin(y)}{y} = 1 - \frac{y^2}{3!} + \frac{y^4}{5!} - \frac{y^6}{7!} \pm \cdots$$

anytime $y \neq 0$.

Can take reciprocal. Get:

$$\left\{ \begin{array}{ll} 1, & y = 0 \\ \\ \dfrac{y}{\sin(y)}, & 0 < y \leq \underline{\dfrac{\pi}{2}} \end{array} \right\}$$

EASY to graph roughly!



is NONZERO and $\underline{C^\infty}$.

OK

Go back to "Piece 1". Note the

$$\left\{ \begin{array}{ll} \dfrac{v/2}{\sin(v/2)}, & 0 < v \leq \pi \\ 1, & v = 0 \end{array} \right\}.$$

$c(v) \geq 1, \ c(0) = 1$

Very Nice

Call this function $c(v)$. It is NONZERO and C^∞. Get:

$$\text{Piece } 1 = \frac{1}{\pi} \int_0^{\pi} \frac{H(x_0+v) - H(x_0+0)}{v} \, c(v) \cdot \sin\left[\left(N+\tfrac{1}{2}\right)v\right] dv \quad *$$

Notice that, up to here, matters hold for any piecewise continuous "starting" $h(x)$ on $[\alpha, \alpha+2\pi]$.

Similarly for Piece 2.

$*$ $\dfrac{\sin(N+\frac{1}{2})v}{v} \to N+\frac{1}{2}$ at $v=0$

Observe that:

$x_0 \equiv$ fixed

$$\frac{H(x_0+v) - H(x_0+0)}{v}\, c(v)$$

$$= \left[H(x_0+v) - H(x_0+0)\right] \frac{c(v)}{v}$$

for $0 < v \leqq \pi$.

$\varepsilon < v \leqq \pi$ if you prefer!!

This expression is (plainly) continuous **except at** the jump discontinuities of $H(x_0+v)$ in $\{0 < v \leqq \pi\}$.

Finite #

What happens as $v \to 0^+$??

Get limit precisely when $H'_+(x_0)$ exists! IE,

$$\lim_{v\to 0^+} \frac{H(x_0+v) - H(x_0+0)}{v} \cdot \lim_{v\to 0^+} c(v).$$

1

KEY POINT.

If $h(x)$ was given to be piecewise C^1 on $[\alpha, \alpha+2\pi]$, then foregoing expression

$$\frac{H(x_0+v)-H(x_0+0)}{v}\cdot \overset{\text{nice}}{c(v)}$$

is thus (at least) piecewise continuous on $[0,\pi]$. Its limiting value as $v \to 0^+$ is just

$$H'(x_0+0)\cdot 1$$

Note! Recall: H is piecewise C^1

So, by R-L lemma,

p. 23 Piece 1 $\longrightarrow 0$ as $N \to \infty$.

Similarly for Piece 2. QED!!

Given FSS:

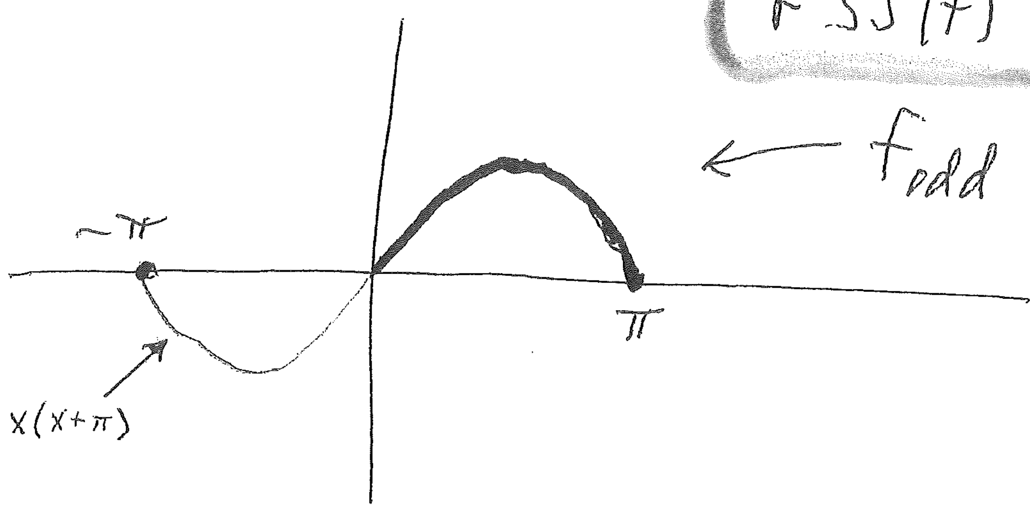
$$x(\pi - x) \sim \frac{8}{\pi} \sum_{n \text{ odd}} \frac{\sin nx}{n^3} \qquad [0, \pi]$$

p. 382

(WHAT IS)

Value of **series** at $-\frac{5}{4}\pi$??

$$\boxed{FSS(f) \equiv FS(f_{odd})}$$



Make **2π**-periodic extension (H) of **f_{odd}**.

$\boxed{\text{H will have no jumps.}}$

$$-\frac{5}{4}\pi + 2\pi = \frac{3}{4}\pi \quad \text{in } [-\pi, \pi]$$

have $H(-\frac{5}{4}\pi) = H(\frac{3}{4}\pi)$

So, get:

$$\underline{\text{Value}} = \frac{3}{4}\pi\left(\pi - \frac{3}{4}\pi\right) = \frac{3}{4} \cdot \frac{1}{4}\pi^2 .$$

$\boxed{\text{ALTERNATE PROBLEM: try } -\frac{9}{4}\pi}$ ANSWER = ??