Key identity from last time:

$$f \sim \sum_{n=1}^{\infty} c_n \varphi_n$$

φ_n orthog on $[a,b]$

$$S_N = \sum_{n=1}^{N} c_n \varphi_n \qquad \text{"} f - S_N \perp S_N \text{"}$$

$$\|f\|^2 = \|f - S_N\|^2 + \|S_N\|^2$$

$$\sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle$$

$$\sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle \leq \|f\|^2, \text{ each } N$$

$$\sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k \rangle \leq \|f\|^2$$

Bessel's inequality

We saw in the preceding lecture that <u>Bessel's</u> <u>inequality</u> becomes an <u>equality</u> for FS and <u>certain</u> kinds of functions.

$\boxed{\text{I. E.}}$

<u>Theorem</u> (Parseval's relation)

Given $f(x)$ <u>of type (abc)</u> on $[a, a+2L]$.
Form $FS(f)$:

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

know $\sum(|A_n| + |B_n|) < \infty$

We then have:

$$\int_a^{a+2L} f(x)^2\,dx = \left(\frac{A_0}{2}\right)^2 \underline{2L} + \sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)\underline{L},$$

corresponding to

$$\langle f, f\rangle = \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k\rangle$$

$\langle c_k\varphi_k, c_k\varphi_k\rangle$

in our old notation.

$$\boxed{\|f\|^2 = \sum_k \|c_k\varphi_k\|^2}$$

See book, p. 207 (8).

A (much) better theorem actually holds!!

Theorem (Parseval's relation for "more f")

Given **any** **piecewise** **continuous** function f on $[a, a+2L]$. Form $FS(f)$:

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

We then have:

$$\int_a^{a+2L} f(x)^2\,dx = \left(\frac{a_0}{2}\right)^2 \underline{2L} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\underline{L}.$$

The proof requires some interesting properties of partial sums $S_N \equiv \sum_{k=1}^{N} c_k \varphi_k$ of "general FS" $f \sim \sum_{k=1}^{\infty} c_k \varphi_k$.

Also uses the fact that we already know Parseval's relation holds for type (abc) functions. "Foot in the door"

Before I get into this, let's do an example or two. These are nice.

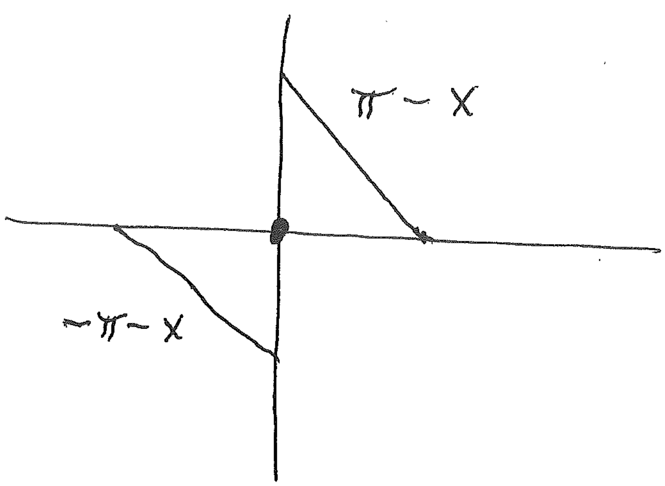
#1

$$\pi - x \sim 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

FSS $[0, \pi]$

p. 382

$$FSS(f) = FS(f_{odd})$$



f_{odd} on $[-\pi, \pi]$

not type abc

$$f_{odd} \sim 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n} \qquad \underline{FS} \quad \text{on } [-\pi, \pi]$$

$$L = \pi$$

$$\int_{-\pi}^{\pi} f_{odd}^2 \, dx = \left(\frac{a_0}{2}\right)^2 \underline{2L} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \underline{L}$$

f_{odd}^2 is even

$$a_n = 0, \; n \geq 0$$
$$b_n = \frac{2}{n}, \; n \geq 1$$

$$2 \int_0^{\pi} (\pi - x)^2 \, dx = \underline{\pi} \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^2$$

$$2 \left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi} = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3}\pi^3 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}$$

Famous Result of Euler ~ 1750

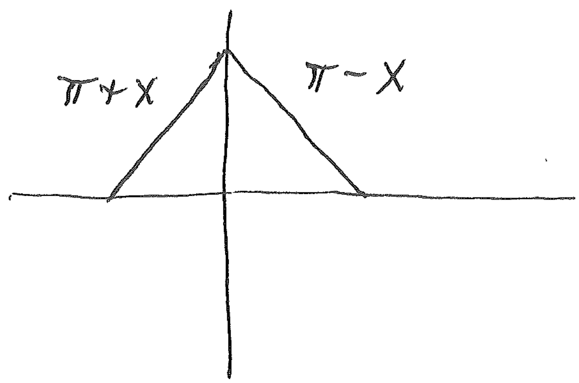
#2

$$\pi - x \sim \frac{\pi}{2} + \frac{4}{\pi} \sum_{\underline{n \text{ odd}}} \frac{\cos nx}{n^2}$$

FCS $[0, \pi]$

p. 381

$FCS(f) = FS(f_{even})$



f_{even} on $[-\pi, \pi]$

clearly $\pi - |x|$

$\boxed{f_{even}: \text{type (abc)}}$

$$f_{even} \sim \frac{\pi}{2} + \frac{4}{\pi} \sum_{\underline{n \text{ odd}}} \frac{\cos nx}{n^2}$$

FS on $[-\pi, \pi]$

$L = \pi$ as before

$a_n = \dfrac{4}{\pi n^2}$ n odd

$$\int_{-\pi}^{\pi} f_{even}^2 \, dx = \left(\frac{\pi}{2}\right)^2 \underbrace{2\pi}_{2L} + \sum_{n \text{ odd}} \left(\frac{4}{\pi n^2}\right)^2 \underbrace{\pi}_{L}$$

$$2\int_0^{\pi} (\pi - x)^2 \, dx = \frac{\pi^2}{2}\pi + \frac{16}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^4} \pi$$

$$2\left[\frac{(\pi-x)^3}{-3}\right]_0^{\pi} = \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{n\ \text{odd}} \frac{1}{n^4}$$

$$\frac{2}{3}\pi^3 = \frac{\pi^3}{2} + \frac{16}{\pi}\sum_{n\ \text{odd}} \frac{1}{n^4}$$

$$\frac{\pi^3}{6} = \frac{16}{\pi}\sum_{n\ \text{odd}} \frac{1}{n^4}$$

$$\boxed{\frac{\pi^4}{96} = \sum_{n\ \text{odd}} \frac{1}{n^4}}$$

Not so well-known!

Let's see if we can't get $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Monday

√ Can try <u>another</u> f. OR, better, <u>use</u> <u>baby</u> <u>algebra</u>.

constructive laziness

Let $S = \sum_{n=1}^{\infty} \frac{1}{n^4}$. Clearly

$$S = \sum_{n \text{ odd}} \frac{1}{n^4} + \sum_{m=1}^{\infty} \frac{1}{(2m)^4}$$

$$S = \frac{\pi^4}{96} + \frac{1}{16} \sum_{m=1}^{\infty} \frac{1}{m^4}$$ Aha!

$$\frac{15}{16} S = \frac{\pi^4}{96} \Longrightarrow$$

$$S = \frac{16}{15} \frac{1}{96} \pi^4 = \frac{\pi^4}{90}$$ OK

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

Famous Result of Euler ~1750

VERY NICE!

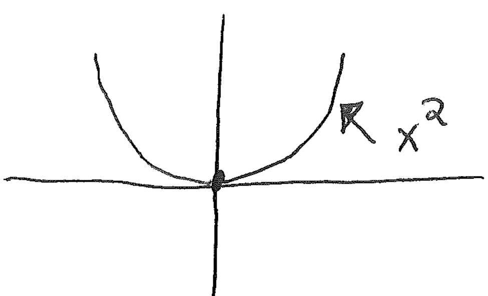
Recall here: the other way of getting nice series is to plug into known FCS, FSS, FS.

p. 381

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) \qquad \begin{array}{c} FCS \\ [0,\pi] \end{array}$$

$$FS(f_{even}) = FCS(f)$$

Draw feven. Trivial! x^2 is $\overbrace{\text{EVEN}}^{\text{(already)}}$!



even 2π-periodic extension

$F_E(x)$

Apply Fourier's Theorem. Get:

$$F_E(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx), \quad \underline{\text{all } x}.$$

E.g., take $x = \pi$.

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\underline{(-1)^n}$$

$$\frac{2}{3}\pi^2 = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}} \quad \text{again!}$$

2 <u>completely</u> different methods.

(VERY) Reassuring.

"MATH IS RIGHT!"

Generalities Again

$\{\varphi_k\}_{k=1}^{\infty}$ orthogonal on $[a, b]$

Assume $\langle \varphi_k, \varphi_k \rangle \neq 0$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

N.B. $\sigma(x)\,dx$ $\quad \sigma > 0$

$$f \sim \sum_{k=1}^{\infty} c_k \varphi_k, \quad c_k = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}$$

special numbers

$$S_N = \sum_{1}^{N} c_k \varphi_k$$

$f - S_N \perp \{\varphi_1, \ldots, \varphi_N\}$

$f - S_N \perp$ all $\sum_{k=1}^{N} t_k \varphi_k$

as before!

$$\boxed{\langle f, f \rangle = \|f - S_N\|^2 + \sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle}$$

$\langle S_N, S_N \rangle$

Fact 1

Given $f(x)$ piecewise continuous on $[a,b]$. Parseval's equation holds for $\langle f,f\rangle \equiv \|f\|^2$ precisely when

$$\lim_{N\to\infty} \|f - S_N\|^2 = 0,$$

i.e.,

$$\lim_{N\to\infty} \|f - S_N\| = 0.$$

Pf

Clear from the box! ∎

Just let $N \to \infty$.

Ie,

$$\|f\|^2 = \lim_{N\to\infty} \|f - S_N\|^2 + \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k \rangle$$

Fact 2 (Least Squares Property) *

Given piecewise continuous $f(x)$ on $[a, b]$. We then have

$$\left\| f - \sum_{k=1}^{N} t_k \varphi_k \right\|^2 = \text{MINIMUM}$$

precisely when $t_k = c_k = \dfrac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}$.

$$\left\{ \begin{array}{l} \text{That minimum will be} \\ \quad \|f\|^2 - \displaystyle\sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle . \end{array} \right\}$$

Pf #1 (by calculus)

I like this one.

$$Q = \left\langle f - \sum_{k=1}^{N} t_k \varphi_k \, , \, f - \sum_{k=1}^{N} t_k \varphi_k \right\rangle$$

For $-\infty < t_j < \infty$. Clearly $Q \geqq 0$.

Good function $Q(t_1, \ldots, t_N)$.

* VERY important

Try to simplify Q. Distributive law again!

$$Q = \langle f, f \rangle - 2\langle f, \sum_{k=1}^{N} t_k \varphi_k \rangle + \langle \sum_{j=1}^{N} t_j \varphi_j, \sum_{k=1}^{N} t_k \varphi_k \rangle$$

$$Q = \langle f, f \rangle - 2\sum_{k=1}^{N} t_k \langle f, \varphi_k \rangle + \sum_{j=1}^{N} \sum_{k=1}^{N} t_j t_k \langle \varphi_j, \varphi_k \rangle$$

0 if $j \neq k$

so,

$$Q = \langle f, f \rangle - 2\sum_{k=1}^{N} t_k \langle f, \varphi_k \rangle + \sum_{j=1}^{N} t_j^2 \langle \varphi_j, \varphi_j \rangle$$

So, Q is just a quadratic function.

$\langle \varphi_j, \varphi_j \rangle > 0$ cf. 10

Clearly there *is* an *absolute* minimum somewhere. Critical Points?

$$\frac{\partial Q}{\partial t_j} = 0 \qquad 1 \le j \le N \qquad \Rightarrow$$

$$-2\langle f, \varphi_j \rangle + 2 t_j \langle \varphi_j, \varphi_j \rangle = 0$$

$$t_j = \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \qquad \text{Aha!}$$

$\boxed{t_j = c_j \text{ is } \underline{\text{sole}} \text{ critical pt}}$

At the CRITICAL POINT, have:

$$-2 t_j \langle f, \varphi_j \rangle + t_j^2 \langle \varphi_j, \varphi_j \rangle \qquad \underline{\text{EACH}}\ \underline{j}$$

$$= -2 \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \langle f, \varphi_j \rangle + \frac{\langle f, \varphi_j \rangle^2}{\langle \varphi_j, \varphi_j \rangle^2} \langle \varphi_j, \varphi_j \rangle$$

$$= -\frac{\langle f, \varphi_j \rangle^2}{\langle \varphi_j, \varphi_j \rangle} = -\left(\frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \right)^2 \langle \varphi_j, \varphi_j \rangle$$

Hence, we get:

$$Q_{min} = \langle f, f \rangle - \sum_{j=1}^{N} c_j^{\circ 2} \langle \varphi_j^{\circ}, \varphi_j^{\circ} \rangle .$$

OK! $\blacksquare$

Pf #2

Start same way, but stay with baby algebra!!

$$Q = \langle f, f \rangle - 2 \sum_{j=1}^{N} t_j^{\circ} \langle f, \varphi_j^{\circ} \rangle + \sum_{j=1}^{N} t_j^{\circ 2} \langle \varphi_j^{\circ}, \varphi_j^{\circ} \rangle$$

Stare at terms with t_j°. $\circledast$ Fixed j

$$-2 t_j^{\circ} \langle f, \varphi_j^{\circ} \rangle + t_j^{\circ 2} \langle \varphi_j^{\circ}, \varphi_j^{\circ} \rangle$$

$$= \langle \varphi_j, \varphi_j \rangle \left[t_j^2 - 2 t_j \frac{\langle f, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} \right]$$

$$= \langle \varphi_j, \varphi_j \rangle \left[t_j^2 - 2 t_j c_j \right]$$

{complete square}

$$= \langle \varphi_j, \varphi_j \rangle \left[(t_j - c_j)^2 - c_j^2 \right]$$

$$\Downarrow$$

$$Q = \langle f, f \rangle - \sum_{j=1}^{N} c_j^2 \langle \varphi_j, \varphi_j \rangle + \sum_{j=1}^{N} \langle \varphi_j, \varphi_j \rangle (t_j - c_j)^2 .$$

It is now 100% obvious where the ABS MIN of Q occurs!! [redacted]

Consider $\{\varphi_k\}_{k=1}^{\infty}$ and $[a,b]$ as before.

One often tries to establish a Parseval's relation for piecewise continuous functions $f(x)$ in this set-up.

kind of

pause and think

KEY OBSERVATION

Let $f(x)$ be piecewise continuous on $[a,b]$. To prove that

$$\lim_{N\to\infty} \|f - S_N\| = 0,$$

it is sufficient to show that, for each $\varepsilon > 0$, we can cook up some linear combination $\sum_{k=1}^{M} u_k \varphi_k$ so that

$$\|f - \sum_{k=1}^{M} u_k \varphi_k\| < \varepsilon.$$

Pf

Notice that $\|f - S_N\|^2$ is decreasing as N increases.

OR: $\|f\|^2 - \sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle$

But,

$$\|f - S_M\|^2 \leq \|f - \sum_{k=1}^{M} a_k \varphi_k\|^2 < \varepsilon^2$$

by least squares property. So, for every $N \geq M$, we have

$$\|f - S_N\|^2 < \varepsilon^2 .$$

I.e., $\|f - S_N\| < \varepsilon$ for all $N \geq M$.

Since $\varepsilon > 0$ was arbitrary, this is equivalent to saying $\lim_{N \to \infty} \|f - S_N\| = 0$.

Small Preparation for Future Lecture.

$[a, b]$ $\quad f(x), \ g(x)$ continuous.

Measures of "distance" between these two functions.

$\boxed{1}$

$$\max_{[a,b]} |f(x) - g(x)| \qquad D$$

$\boxed{2}$ Can also use average distance

$$\frac{1}{b-a} \int_a^b |f(x) - g(x)| \, dx \qquad D_1$$

$\boxed{3}$ Root Mean Square Distance

$$\sqrt{\frac{1}{b-a} \int_a^b |f(x) - g(x)|^2 \, dx} \qquad D_2$$

$$= \frac{1}{\sqrt{b-a}} \|f - g\|$$

Basic Fact

$$D_1 \leqq D_2 \leqq D \qquad D_\infty$$

Pf

$D_2 \leqq D$ obvious by plug in.

$D_1 \leqq D_2$?? Need:

$$\frac{1}{b-a}\int_a^b |f(x)-g(x)|\,dx \leqq \frac{1}{\sqrt{b-a}}\|f-g\|$$

OR

$$\int_a^b |f(x)-g(x)|\,dx \leqq \sqrt{b-a}\,\|f-g\|$$

Use Cauchy-Schwarz inequality

$$\int_a^b |f-g|\cdot 1\,dx \leqq \underbrace{\sqrt{\int_a^b |f-g|^2 dx}}_{\|f-g\|}\ \underbrace{\sqrt{\int_a^b 1^2 dx}}_{\sqrt{b-a}}$$

QED ■