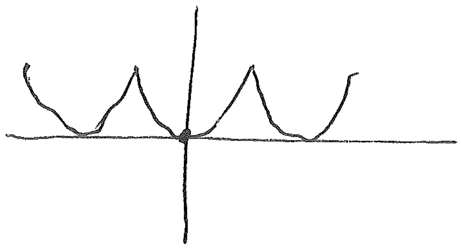
Practice with another self-consistency

$f(x) = x^2$, $[0, \pi]$ FCS p. 381



$F_E(x) =$ easy

$$F_E(x) = \frac{\pi^2}{3} + 4 \sum_1^\infty \frac{(-1)^n}{n^2} \cos(nx) \quad , \quad \text{all } x$$

Try $F_E(0) = 0$. Gives:

$$0 = \frac{\pi^2}{3} + 4 \sum_1^\infty \frac{(-1)^n}{n^2}$$

so

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$ *yes!*

Let: $\mathcal{O} = \sum_{n \text{ odd}} \frac{1}{n^2}$, $\mathcal{E} = \sum_{n \text{ even}} \frac{1}{n^2}$.

$$\frac{\pi^2}{12} = \mathcal{O} - \mathcal{E}$$

$$\frac{\pi^2}{12} = \mathcal{O} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2}$$

$$\frac{\pi^2}{12} = \mathcal{O} - \frac{1}{4}\left[\mathcal{O} + \mathcal{E}\right]$$

$$\frac{\pi^2}{12} = \frac{3}{4}\mathcal{O} - \frac{1}{4}\mathcal{E}$$

$$\frac{\pi^2}{3} = 3\mathcal{O} - \mathcal{E}$$

$$\begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$$

Baby Algebra $\Longrightarrow$

$$\mathcal{O} = \frac{\pi^2}{8}, \quad \mathcal{E} = \frac{\pi^2}{24}. \quad \checkmark\checkmark$$

So,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \mathcal{O} + \mathcal{E} = \pi^2 \left[\frac{1}{8} + \frac{1}{8 \cdot 3} \right]$$

$$= \pi^2 \frac{1}{8} \left[\frac{4}{3} \right]$$

$$\boxed{\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$ As Before!

NICE!

We saw, in last lecture, 2/10, that, to prove Parseval's Equation for <u>any</u> <u>piecewise</u> <u>continuous</u> fcn f,

[where $f \sim \sum_{n=1}^{\infty} c_n \varphi_n$ $\quad I = [a, b]$

$S_N = \sum_{k=1}^{\infty} c_k \varphi_k$]

we must prove

Equivalent

$$\lim_{N \to \infty} \| f - S_N \| = 0 .$$

Then, we made a KEY OBSERVATION

p. 17 there!!

(that uses least squares property of "c_n").

Consider $\{\varphi_k\}_{k=1}^{\infty}$ and $[a,b]$ as before.

One often tries to establish a Parseval's relation for piecewise continuous functions $f(x)$ in this $\underbrace{\ }_{\text{kind of}}$ set-up.

pause and think

KEY OBSERVATION

Let $f(x)$ be piecewise continuous on $[a,b]$. To prove that

$$\lim_{N\to\infty} \|f - S_N\| = 0,$$

it is sufficient to show that, for each $\varepsilon > 0$, we can cook up some linear combination $\sum_{k=1}^{M} u_k \varphi_k$ so that

$$\|f - \sum_{k=1}^{M} u_k \varphi_k\| < \varepsilon.$$

$I = [\alpha, \alpha + 2L]$ ✓✓

$\{\varphi_k(x)\} \longleftrightarrow \left\{\cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right\}$ list

orthogonal

$$\langle g, v \rangle = \int_\alpha^{\alpha+2L} g(x)\, v(x)\, dx$$

Important

Theorem (Completeness thm) ← for concept $3\frac{1}{2}$

Let $f(x)$ be piecewise continuous on I.
Let $\varepsilon > 0$. We can then cook up
some linear combination $\sum_{k=1}^{M} u_k \varphi_k$ so
that

$$\left\| f - \sum_{k=1}^{M} u_k \varphi_k \right\| < \varepsilon .$$

Compare here "KEY OBSERVATION" in preceding lecture (page 17).

Once we <u>prove</u> this Theorem, we thus know

$$\lim_{N\to\infty} \|f - S_N\| = 0 ,$$

where S_N = partial sum of $FS(f)$.

To be perfectly clear here, quite generally,

$$f \sim \sum_{k=1}^{\infty} c_k \varphi_k(x) \quad , \quad c_k = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}$$

$$S_N(x) \equiv \sum_{k=1}^{N} c_k \varphi_k(x) \quad \Longrightarrow$$

$$\|f\|^2 = \|f - S_N\|^2 + \sum_{k=1}^{N} c_k^2 \langle \varphi_k, \varphi_k \rangle$$

$$\|f - S_N\| \to 0 \iff \|f\|^2 = \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k \rangle$$

$\boxed{\text{p. 10}}$ previous lecture

Parseval's equation

PLAN or ROAD MAP:

T = "TRIG ← some big lake

finite # of

$\|f - g\|$ = distance f to g

(1) "abc" is close to T (within ε)

(2) "step function" is close to "abc"

[hence close to T]

(3) "piecewise continuous" is close to "step"

[hence close to "abc"
hence close to T]

SO:

"piecewise continuous" is close to T.

QED!! DONE!! ■

I will give a partial proof of Theorem.
This will be enough for us.

(The full proof requires <u>uniform</u> continuity of continuous functions on $[a,b]$.)

Prelim #1

If $f(x)$ is of type <u>(abc)</u> on I, things are OK.

Pf

Know here: $\boxed{x \in I}$

$$f(x) = \frac{1}{2}A_0 + \sum_{1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty}\left(|A_n| + |B_n|\right) < \infty$$

Weierstrass M_n-test, $M_n = |A_n| + |B_n|$

dominated + uniform convergence

$$\left| F(x) - \tfrac{1}{2}A_0 - \sum_{n=1}^{N} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right) \right|$$

$$< \frac{\varepsilon}{\sqrt{2L}} \quad \text{for } N \geqq N_\varepsilon .$$

Consider

$$\sum_{k=1}^{M} u_k \varphi_k \longleftrightarrow \tfrac{1}{2}A_0 + \sum_{1}^{N} \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} \right)$$

with, say, $N = N_\varepsilon + 1$. Get:

$$\int_I \left| F(x) - \sum_1^M u_k \varphi_k \right|^2 dx$$

$$< \int_I \frac{\varepsilon^2}{2L}\, dx = \varepsilon^2$$

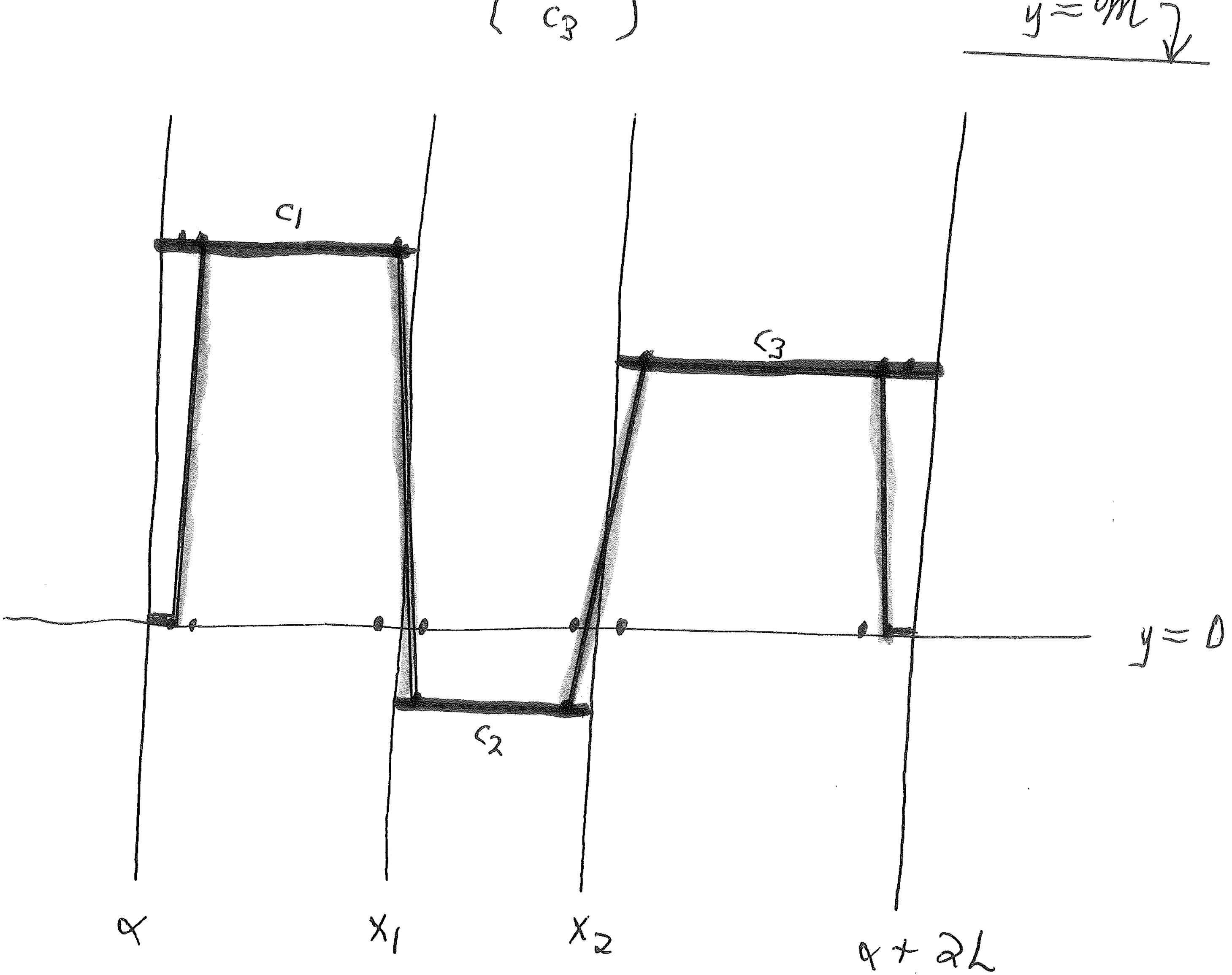
so

$$\left\| F - \sum_1^M u_k \varphi_k \right\| < \varepsilon .$$ OK ∎

Before I discuss Prelim #2, it's best to consider an <u>example</u> of a **piecewise constant** function on I.

$\boxed{\delta = \text{TINY}}$

E.G. $\quad f(x) = \left\{ \begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix} \right\}$



$$\left\{ \begin{matrix} \alpha+\delta,\ \alpha+2\delta \\ x_1-\delta,\ x_1+\delta \quad \text{and} \quad x_2-\delta,\ x_2+\delta \\ \alpha+2L-2\delta,\ \alpha+2L-\delta \end{matrix} \right\}$$

$y = -M$

Let $\ell(x)$ be the CONTINUOUS piecewise linear graph defined via

black/green lines (two being 0), c_1, c_2, c_3.

By inspection,

$$f(x) - \ell(x) = 0 \quad \text{except on} \quad \begin{array}{l} [a, a+2\delta] \\ [x_1-\delta, x_1+\delta] \\ [x_2-\delta, x_2+\delta] \\ [a+2L-2\delta, a+2L] \end{array}$$

and, very crudely,

$$|f(x) - \ell(x)| \lesssim 2M \quad \text{on the exceptional intervals}.$$

So:

$$\int_I |f-\ell|^2 dx = \int_a^{a+2\delta} + \int_{x_1-\delta}^{x_1+\delta} + \int_{x_2-\delta}^{x_2+\delta}$$

$$+ \int_{a+2L-2\delta}^{a+2L}$$

$$\lesssim (2M)^2 2\delta + (2M)^2 2\delta + (2M)^2 2\delta$$
$$+ (2M)^2 2\delta$$

$$= 32 M^2 \delta$$

Legitimate!

Freeze your $\delta < \dfrac{\varepsilon^2}{32M^2}$. Get:

$$\|f - \ell\|^2 < \varepsilon^2$$

$$\|f - \ell\| < \varepsilon .$$

piecewise constant

Thus: $f(x)$ can always be approximated in the $\|f_1 - f_2\|$ sense by a CONTINUOUS piecewise linear function which is identically 0 near the endpoints of $I = [a, a + 2L]$.

very useful

On p. 5 above, we had "FS"

$I = [a, a+2L]$, $\{\varphi_k\} \leftrightarrow \{\cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\}$.

a "step function"

Let $f(x)$ be piecewise constant on I. Let $\varepsilon > 0$. We showed that there exists a type (abc) function $\ell(x)$ on I such that

$\ell(x)$ is piecewise linear on I;

$\ell(x) \equiv 0$ near endpts of I;

$\|f(x) - \ell(x)\| < \varepsilon$.

(Recall our "beautiful picture"!!)

Prelim #2 (an application of "abc")

Let $f(x)$ be any piecewise constant function on I. Let $\varepsilon > 0$. We can then cook up some linear combination $\sum_1^M u_k \varphi_k$ so that

$$\|f(x) - \sum_1^M u_k \varphi_k\| < \varepsilon.$$

Pf of Prelim #2

WE WILL USE PRELIM #1.

Select piecewise linear $\ell(x)$ as above on I ← as in picture
so that

$$\| f - \ell \| < \frac{\varepsilon}{2}.$$

Apply Prelim #1 since ℓ is type (abc). ✓✓
Get some linear combination $\sum_1^M u_k \varphi_k$
so that

$$\| \ell - \sum_1^M u_k \varphi_k \| < \frac{\varepsilon}{2}.$$

Recall Minkowski's inequality!!

$$\| g_1 + g_2 \| \leq \| g_1 \| + \| g_2 \|$$

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problem 7

Write:

$$f - \sum_1^M u_k \varphi_k = (f - \ell) + (\ell - \sum_1^M u_k \varphi_k).$$

Get:

$$\| f - \sum_1^M u_k \varphi_k \| \leq \left(\text{less than } \frac{\varepsilon}{2}\right) + \left(\text{less than } \frac{\varepsilon}{2}\right)$$

$$< \varepsilon.$$ QED ∎

Prelim #3

Let $g(x)$ be any piecewise continuous function on I. Let $\varepsilon > 0$. We can always find some piecewise constant function $h(x)$ on I so that $\|g-h\| < \varepsilon$.

PF

Very similar to our proof $\left(\begin{smallmatrix} 1/29 \\ 13 \; 22 \end{smallmatrix}\right)$ of R-L lemma. Just need to do some fancy footwork. (One uses the uniform continuity property of every continuous function $g(x)$ on $[A, B]$.)

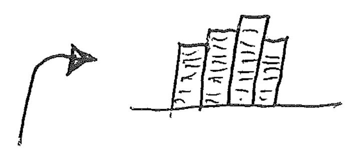
I'll show you the basic idea! Anyone interested can fill in full details.

5%?

Assume $g(x)$ has 3 "continuous chunks".

g_1, g_2, g_3 on I_1, I_2, I_3

Use RIEMANN SUM IDEA for ⊛ g_1, g_2, g_3 plus uniform continuity of each g_j.



Get piecewise constant $h_j(x)$ on I_j so that

$$|h_j(x) - g_j(x)| < 1 \quad \text{on } I_j$$

$$\int_{I_j} |h_j(x) - g_j(x)|\, dx < \frac{\varepsilon^2}{3}.$$

No Problem! But, now,

$$|h_j(x) - g_j(x)|^2 \leq |h_j(x) - g_j(x)|.$$

⊛ SECRET: can get by without!!

So,

$$\int_{I_j} |h_j(x) - g_j(x)|^2 dx < \frac{\varepsilon^2}{3}.$$

Let $h(x)$ = the **piecewise constant** fcn obtained by combining h_1, h_2, h_3 on $I_1 \cup I_2 \cup I_3 = I$.

Clearly:

$$\int_I |h(x) - g(x)|^2 dx = \sum_{j=1}^{3} \int_{I_j} |h_j(x) - g_j(x)|^2 dx$$

$$< \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} = \varepsilon^2$$

So

$$\|g - h\| < \varepsilon.$$

QED $\blacksquare$

Alas, it is now very easy to finish the proof of the "Completeness Theorem" (See 5.).

Go slow! Given $f(x)$ **piecewise continuous** on I. Given $\varepsilon > 0$. Cook up some **piecewise constant** function $h(x)$ on I so that

Riemann Sum Idea!

$$\|f - h\| < \frac{\varepsilon}{2} \quad \text{(Prelim \#3 on 14)}.$$

p. 12 (to h)

Apply Prelim #2. Can cook up some $\sum_1^M u_k \varphi_k$ so that, for $h(x)$,

$$\left\| h(x) - \sum_1^M u_k \varphi_k \right\| < \frac{\varepsilon}{2}.$$

But,

$$f - \sum_1^M u_k \varphi_k = (f - h) + \left(h - \sum_1^M u_k \varphi_k\right)$$

and $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ (Minkowski).

Get:

$$\| f - \sum_{1}^{M} \underline{u_k} \varphi_k \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

That's what we sought! QED

As a SUMMARY:

Completeness Theorem said we could get

$$\| f - \sum_{1}^{M} \underline{u_k} \varphi_k \| < \varepsilon$$

{see p. 5}

{some M, $\underline{u_k}$}.

It $\Downarrow$ by KEY OBSERVATION Prev. Lec. p. 17 2/10

$$\boxed{\lim_{N \to \infty} \| f - S_N \| = 0}$$

!!

$\Downarrow$ by p. 10 previous Lec. 2/10 OR p. 6 above!!

$$\langle f, f \rangle = \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k \rangle$$

Parseval

$$\boxed{\| f \|^2 = \sum_{k=1}^{\infty} \| c_k \varphi_k \|^2}$$

for case $3\frac{1}{2}$

$$f \sim \sum_{k=1}^{\infty} c_k \varphi_k$$

$\Uparrow$

Vital to note that, for our "completeness theorem" in regard to $\left\{\cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right\}$ and $[\alpha, \alpha+2L]$, we <u>have</u> <u>exploited</u> <u>KEY</u> properties of type (abc) functions and their FS on $[\alpha, \alpha+2L]$. $\checkmark\checkmark\checkmark$

(As I promised.)

NOTE —
Currently, we only have Parseval's relation for <u>FS</u> on $[\alpha, \alpha+2L]$.

FORTUNATELY, by a <u>trick</u>, we can then (easily) get Parseval for

<u>FSS</u> $\left\{\sin\frac{n\pi x}{L}\right\}$ on $[0, L]$;

<u>FCS</u> $\left\{\cos\frac{m\pi x}{L}\right\}$ on $[0, L]$.

Theorem (Parseval for FSS and FCS)

Given any piecewise continuous function $f(x)$ on $[0,L]$. Form $FSS(f)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{on } [0,L].$$

We then have:

$$\int_0^L f(x)^2\,dx = \sum_{n=1}^{\infty} b_n^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} b_n^2 \left(\frac{L}{2}\right).$$

Similarly, form $FCS(f)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{on } [0,L].$$

We then have:

$$\int_0^L f(x)^2\,dx = \left(\frac{a_0}{2}\right)^2 \int_0^L 1\,dx + \sum_{n=1}^{\infty} a_n^2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \left(\frac{a_0}{2}\right)^2 L + \sum_{n=1}^{\infty} a_n^2 \left(\frac{L}{2}\right).$$