M 2/15 ⓪

Agenda

1. Completeness — definition (205 top thm 2)

2. Parseval for FSS, FCS cf. 208(2)(3)

3. Completeness — literally

4. Better form of Fourier's Thm (with **uniform** conv) (not in book)

5. Classical integration/diff thms (50 2/3 down)

6. Integration of **Fourier** Series (56(3) 58(top))
   {specifically as a type of series}

$\{\varphi_n\}_{n=1}^{\infty}$ orthog $[a,b]$

$\sigma(x)\,dx$

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx$$

$$f \sim \sum_{n=1}^{\infty} c_n \varphi_n \,, \qquad c_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle}$$

$$S_N = \sum_{n=1}^{N} c_n \varphi_n$$

$$\left\{ \begin{array}{l} \|f\|^2 = \|f - S_N\|^2 + \|S_N\|^2 \,, \qquad f - S_N \perp S_N \\[1em] \|f\|^2 = \|f - S_N\|^2 + \sum_{n=1}^{N} c_n^2 \langle \varphi_n, \varphi_n \rangle \end{array} \right.$$

Least **squares** (interpretation of c_n)

$$\|f - S_N\|^2 = \min \Big\|f - \sum_{n=1}^{N} t_n \varphi_n \Big\|^2 .$$

GET $\overline{\lim_{N \to \infty}} \|f - S_N\| = 0 \iff \|f\|^2 = \sum_{n=1}^{\infty} c_n^2 \langle \varphi_n, \varphi_n \rangle$

$\iff$ for each $\varepsilon > 0$ can find <u>some</u>

$\sum_{k=1}^{M} u_k \varphi_k$ SO THAT

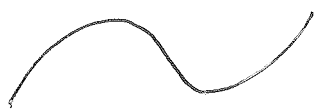
$$\Big\|f - \sum_{k=1}^{M} u_k \varphi_k \Big\| < \varepsilon$$

We say $\{\varphi_n\}_{n=1}^{\infty}$ is complete

when these 3 (equiv.) conditions are

valid for every piecewise continuous

f on $[a, b]$.



We have proved that

$$\left\{\cos\frac{n\pi x}{L}\right\}_{n=0}^{\infty} \cup \left\{\sin\frac{m\pi x}{L}\right\}_{m=1}^{\infty}$$

is complete on $[\alpha, \alpha + 2L]$.

"FS"

concept $3\frac{1}{2}$

Theorem (Parseval for FSS and FCS)

Given any piecewise continuous function $f(x)$ on $[0, L]$. Form FSS(f):

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{on } [0, L].$$

We then have:

$$\int_0^L f(x)^2 dx = \sum_{n=1}^{\infty} b_n^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \sum_{n=1}^{\infty} b_n^2 \underline{\left(\frac{L}{2}\right)}.$$

Similarly, form FCS(f):

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad \text{on } [0, L].$$

We then have:

$$\int_0^L f(x)^2 dx = \left(\frac{a_0}{2}\right)^2 \int_0^L 1^2 dx + \sum_{n=1}^{\infty} a_n^2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \left(\frac{a_0}{2}\right)^2 \underline{L} + \sum_{n=1}^{\infty} a_n^2 \underline{\left(\frac{L}{2}\right)}.$$

Proof

I'll do FSS. FCS is similar.

Given f on $[0,L]$. Form FSS(f) as above. Form $f_{odd}(x)$ on $[-L,L]$. Remember that $f_{odd}(0)=0$. The fcn $f_{odd}(x)$ is piecewise continuous on $[-L,L]$. Know $FSS(f) \equiv FS(f_{odd})$.

Hence, as FS on $[-L,L]$,

$$f_{odd}(x) \sim \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Apply Parseval for FS. Get

$$\int_{-L}^{L} f_{odd}^2\,dx = \sum_{1}^{\infty} b_n^2 \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$2\int_0^L f^2\,dx = \sum_{1}^{\infty} b_n^2\, 2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx.$$

Just divide by 2! Done!

QED

SLICK!

Just for clarity, Parseval for $\underline{FS(f)}$ on $[\alpha, \alpha+2L]$ gave us:

$$\int_{\alpha}^{\alpha+2L} f^2\,dx = \left(\frac{a_0}{2}\right)^2 \underline{2L} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\,\underline{L}$$

for

$$f \sim \frac{a_0}{2} + \sum_{1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Also useful to recall that for $\underline{\underline{ANY}}$ orthogonal "family" $\{\varphi_k\}_{k=1}^{\infty}$ on $[a, b]$ wherein

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,dx, \qquad \boxed{\sigma(x)\,dx} \quad \sigma(x) > 0$$

one has Parseval's equation

$$\langle f, f\rangle = \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k\rangle$$

$\underline{\text{if and only if}}$ $\lim\limits_{N\to\infty} \|f - S_N\| = 0.$

(based on this)

So, there is now <u>no</u> question that for <u>FS</u>, <u>FSS</u>, <u>FCS</u>, <u>we do have</u>

$$\lim_{N\to\infty} \|f - S_N\| = 0$$

for <u>every</u> piecewise continuous $f(x)$ on the relevant interval.

WE WILL USE.

Format

$$\frac{1}{2}a_0 + \sum_{n=1}^{N} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L}$$

$$\frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos\frac{n\pi x}{L}$$

partial sum

$\{\varphi_k\}_{k=1}^{\infty}$, orthogonal "family", $[a,b]$

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,dx \qquad \boxed{\sigma(x)dx}$$

"Completeness theorem" officially spoke of

$$\|f - \sum_{1}^{M} \underline{u_k}\varphi_k\| < \varepsilon \qquad \left(\begin{array}{l}\text{any}\\ \text{piecewise continuous}\\ f\end{array}\right)$$

for **some** M, u_k.

Page $\frac{1}{2}$ (then) gave (us) the 3-way equivalence!

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as GENERAL BACKGROUND:

HAVE Another reason why we say $\{\varphi_k\}_{k=1}^{\infty}$ has the completeness property.

$\boxed{\text{Very Natural!!}}$

Namely: you **can't** make the **orthogonal** "family" bigger!!

$$\{\varphi_1, \varphi_2, \ldots\} + \{\psi\} \quad ??$$

To still be orthogonal, would **need**

$$\langle \psi, \varphi_k \rangle = 0, \ \text{all } k. \implies \boxed{c_k(\psi) = 0}$$

To **avoid** TRIVIA, also want:

$$\langle \psi, \psi \rangle = \int_a^b \psi^2 dx > 0.$$

Use 3-way equivalence on p. 1/2.

Completeness in $\{\varphi_k\}_{k=1}^{\infty}$ means **Parseval** holds. Yes, but...

Apply Parseval to $\langle \psi, \psi \rangle$. Get:

$$\langle \psi, \psi \rangle = \sum_{k=1}^{\infty} 0^2 \langle \varphi_k, \varphi_k \rangle = 0.$$

$$\boxed{\text{CONTRADICTION!}}$$

As an example, consider

$$\varphi_k(x) = \sin(k+1)x \ , \quad k \geq 1$$

on $[0,\pi]$. $\boxed{\langle u,v\rangle = \int_0^\pi u(x)v(x)\,dx}$

Have:

$$\{\varphi_k\}_{k=1}^{\infty} = \{\sin 2x, \sin 3x, \ldots\}.$$

Can this family be complete? NO!

If it were complete, you'd have Parseval

$$\langle f,f\rangle = \sum_{k=1}^{\infty} c_k^2 \langle \varphi_k, \varphi_k\rangle$$

for all piecewise continuous f on $[0,\pi]$.

Take $f(x) = \sin x$. Plug in.

$$\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}\ ; \quad \langle f,f\rangle = \frac{\pi}{2}\ ;$$

$$\int_0^\pi \sin x \cdot \sin(k+1)x\,dx = 0 \Rightarrow c_k = 0.$$

CONTRADICTION!

It's very important for our later work with separation of variables [in the textbook] to make sure that, when we start obtaining orthogonal "families", we do not stop too early; i.e., that we strive to ensure that "nothing gets omitted".

Better Form of Fourier's Theorem

Let $f(x)$ be piecewise C^1 on $[\alpha, \alpha+2L]$. Let $F(x)$ be the $2L$-periodic extension of $f(x)$. Form $FS(f)$:

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

{Keep cos ~ sin paired.} Let $[x_1, x_2]$ be any closed interval on which $F(x)$ can be seen to be continuous (i.e. no jumps). Then,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

converges uniformly on the closed interval $[x_1, x_2]$ to $F(x)$.

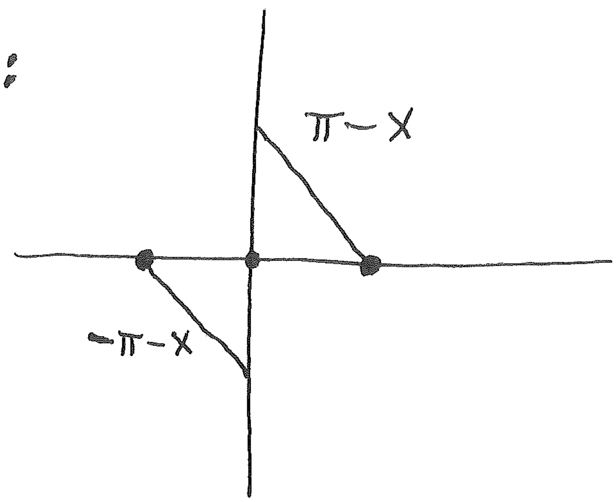
Proof #1

By a slightly long-winded review of our step-by-step proof of the R-L lemma and the <u>original</u> form of Fourier's Theorem. ■ Exploits uniform continuity!

Proof #2

Using a trick. ■

Hint:



Supplement (by scrutinizing the proof)

There exists an $\mathfrak{M} > 0$ so that

$$|S_N(x)| \leqq \mathfrak{M}$$

for <u>all</u> $N \geqq 1$ and $x \in \mathbb{R}$.

$\mathfrak{M}$ depends on f

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Earlier, on the matter of integration, we had the following basic theorem!

VERY USEFUL!!

Thm (classical integration thm)

Let $S(x) = \sum_{n=1}^{\infty} c_n(x)$ be uniformly conv on $[a, b]$. Assume that $c_n(x)$ is continuous on $[a, b]$. Let $p(x)$ be piecewise continuous on $[a, b]$. Then:

(i) $S(x)$ is continuous on $[a, b]$;

(ii) $\int_a^x p(t) S(t)\, dt = \sum_1^{\infty} \int_a^x p(t) c_n(t)\, dt$

for $a \leq x \leq b$;

(iii) we have uniform conv on $[a, b]$ in assertion (ii).

NICE!

Note: can also treat $\int_{x_0}^{x} p(t) S(t)\, dt$ with $a < x_0 < b$.

$$\boxed{\int_a^x - \int_a^{x_0}}$$

One wonders about differentiation!

$$\left\{ \text{for } \sum_{n=1}^{\infty} c_n(x) \right\}$$

Reminder:

Let $h(x)$ be C^1 on $[a,b]$. Then: $h'(x)$ is continuous on $[a,b]$ with the convention that $h'(a)$ and $h'(b)$ refer to 1-sided derivatives.

N.B.

$$\boxed{\begin{array}{l} |x| \text{ is } C^1 \text{ on } [-1,0] \\ |x| \text{ is } C^1 \text{ on } [0,1] \\ \text{but } |x| \text{ is } \underline{\text{not}}\ C^1 \text{ on } [-1,1] \end{array}}$$

Be Careful!

book p. 50 (2/3 down)

classical

THEOREM (differentiation thm)

Given $S(x) = \sum_{n=1}^{\infty} c_n(x)$ on $[a,b]$ with terms $c_n(x)$ which are C^1. Assume that series $S(x)$ is pointwise convergent on $[a,b]$. Assume further that the series

$$\underline{T(x)} \equiv \sum_{n=1}^{\infty} c_n'(x)$$

is uniformly convergent on $[a,b]$.

Then:

(i) $S(x)$ is C^1 on $[a,b]$; and in fact

(ii) $S'(x) = T(x)$ on $[a,b]$.

Proof

At the outset, we do $\boxed{\text{not}}$ even know that $S(x)$ is continuous.

Know that $c_n'(x)$ is continuous on $[a,b]$ for EACH n. By <u>uniform</u> conv, $T(x)$ is continuous. Can apply <u>integration</u> thm to T. Get:

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$$\int_a^x T(t)\,dt = \sum_{n=1}^{\infty} \int_a^x c_n'(t)\,dt$$

$$= \sum_{n=1}^{\infty} \left[c_n(x) - c_n(a) \right]$$

$$= S(x) - S(a).$$

Hence,

$$S(x) = S(a) + \int_a^x T(t)\,dt, \quad a \leq x \leq b.$$

Apply fund theorem of integral calc!!!

Deduce that $S(x)$ is continuous on $[a,b]$ <u>and</u> <u>that</u>

$$S'(x) = T(x), \quad a \leq x \leq b.$$

As such, $S(x)$ is patently C^1 on $[a,b]$. ∎

Important Remark.

The differentiation theorem is often supplemented with a third assertion; namely,

(iii) the series $S(x) = \sum_{n=1}^{\infty} c_n(x)$ is automatically UNIFORMLY CONV on $[a,b]$.

Indeed, when we apply the integration thm to series $T(x) \equiv \sum_{1}^{\infty} c_n'(x)$, we find that

$$\int_a^x T(t)\,dt = \sum_{n=1}^{\infty} [c_n(x) - c_n(a)]$$

is UNIF CONV on $[a,b]$. But, fixed series $\sum_{n=1}^{\infty} c_n(a)$ is trivially unif conv on $[a,b]$. As such, can now just add to get $\sum_{n=1}^{\infty} c_n(x)$ unif conv on $[a,b]$. OK

Easy Example.

Consider the function defined by

$$S(x) = \sum_{n=1}^{\infty} \frac{2}{n^3} \sin(nx)$$

for $-\pi \leq x \leq \pi$. The S-series is clearly convergent. Look at the "differentiated series"

$$\sum_{n=1}^{\infty} \frac{2n}{n^3} \cos(nx)$$

$$= \sum_{n=1}^{\infty} \frac{2}{n^2} \cos(nx).$$

Take $M_n = \frac{2}{n^2}$. This series is UNIF conv on $[-\pi, \pi]$. Thus, whatever $S(x)$ is, we know it is C^1 and that

$$S'(x) = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos(nx).$$

Yes!

especially repeated

Differentiation of FS can be tricky / sticky.

but **Integration of FS** is **trivial** thanks to the following theorem.

EASY

Compare p. 56 (3). ***

THM

Given $[\alpha, \alpha + 2\pi]$; let $f(x)$ be **piecewise continuous** on $[\alpha, \alpha + 2\pi]$. Let $F(x)$ be the 2π-periodic extension of $f(x)$ {with possible adjustment at $x = \alpha + 2\pi k$}. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

on $[\alpha, \alpha + 2\pi]$. (**We do NOT know or care if equality holds.**) We then have:

$$\int_\alpha^x f(t)\,dt = \frac{a_0}{2}\int_\alpha^x 1\,dt + \sum_{1}^{\infty}\left(a_n \int_\alpha^x \cos nt\,dt + b_n \int_\alpha^x \sin nt\,dt\right)$$

for $\{\alpha \leq x \leq \alpha + 2\pi\}$, with **uniform** convergence for such x-values. To go **outside** $[\alpha, \alpha + 2\pi]$,

simply replace the LHS by $\int_a^x F(t)\,dt$.
Uniform convergence again holds provided
x is kept bounded. Similarly for $[a, a+2L]$
and $(2L)$-periodic.

Proof (NOT so easy!)

Let $S_N(x) = \frac{a_0}{2} + \sum_1^N (a_n \cos nx + b_n \sin nx)$ be
the usual partial sum. We know that $S_N(t)$
is 2π-periodic. So is $F(t)$. We also know
that Parseval's equation (and the related
completeness theorem) is valid on $[a, a+2\pi]$.
Hence,

$$\lim_{N\to\infty} \|f - S_N\|^2 = 0$$

on $[a, a+2\pi]$. By periodicity, we then get

$$(*) \quad \lim_{N\to\infty} \int_a^{a+2\pi Q} |F(t) - S_N(t)|^2\,dt = 0$$

for each integer $Q \geq 1$. Keep $x \in [a, a+2\pi Q]$.
EG $Q \approx$ giant.

Notice that:

$$\int_a^x F(t)\,dt - \left(\frac{a_0}{2}\right)\int_a^x 1\,dt - \sum_1^N \left(a_n \int_a^x \cos nt\,dt + b_n \int_a^x \sin nt\,dt\right)$$

$$= \int_a^x F(t)\,dt - \int_a^x S_N(t)\,dt$$

$$= \int_a^x \left[\underline{F(t) - S_N(t)}\right] dt .$$

Hence,

$$\left| \int_a^x F(t)\,dt - \int_a^x S_N(t)\,dt \right|$$

$$\leqq \int_a^x |F(t) - S_N(t)|\,dt$$

$$\leqq \int_a^{a+2\pi Q} |F(t) - S_N(t)|\,dt$$

by (C–S)

[redacted]

$$\leqq \sqrt{\int_a^{a+2\pi Q} 1^2\,dt}\ \sqrt{\int_a^{a+2\pi Q} |F - S_N|^2\,dt}$$

$$= \sqrt{2\pi Q}\ \sqrt{\int_a^{a+2\pi Q} |F(t) - S_N(t)|^2\,dt} .$$

Select N_ε so big that

$$\int_a^{a+2\pi Q} |F(t) - S_N(t)|^2\, dt < \frac{\varepsilon^2}{2\pi Q}$$

holds for all $N \geq N_\varepsilon$. This *is* possible by (*) above. We promptly get:

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$$\left| \int_a^x F(t)\,dt - \left(\frac{a_0}{2}\right)\int_a^x 1\,dt - \sum_{n=1}^{N}\left(a_n \int_a^x \cos nt\,dt + b_n \int_a^x \sin nt\,dt\right)\right|$$

$$< \varepsilon$$

for **every** $N \geq N_\varepsilon$ and $x \in [a, a+2\pi Q]$.

This proves that

$$\int_a^x F(t)\,dt = \left(\frac{a_0}{2}\right)\int_a^x 1\,dt + \sum_{n=1}^{\infty}\left(a_n \int_a^x \cos nt\,dt + b_n \int_a^x \sin nt\,dt\right)$$

holds on $[a, a+2\pi Q]$ **with UNIFORM conv.**

The reasoning for $[\alpha - 2\pi Q, \alpha]$ is similar. Likewise for doing the counterpart with $2\pi \Longrightarrow 2L$. ■

Can get many interesting, even subtle, series this way!

(VERY) Useful Remark

An analogous term-by-term integration theorem holds for $\int_\alpha^x p(t)F(t)\,dt$, where $p(t)$ is (say) any piecewise continuous function on $\mathbb{R}$. [The function $p(t)$ need not be periodic.]

One checks this "augmented" theorem by making easy changes in the above proof (wherein $p(t) \equiv 1$). Note that the term $\int_\alpha^{\alpha+2\pi Q} 1^2\,dt$ simply becomes

$$\int_\alpha^{\alpha+2\pi Q} p(t)^2\,dt.$$