Agenda

(A) Harder Example on Diff. of a FS

(B) FS(F) , F = 2π-periodic ext of f

(C) "Circular Thm"

(D) Traditional Thm on Diff. of FS

(E) Abel's Lemma / Dirichlet's TEST

(F) Uniform Conv of Certain FS

(G) Baby Stuff

A

In previous lecture, we had

$$S(x) = \sum_{n=1}^{\infty} \frac{2}{n^3} \sin(nx)$$

on $-\pi \leq x \leq \pi$. We saw by classical differentiation thm that

$$S'(x) = \sum_{n=1}^{\infty} \frac{2}{n^2} \cos(nx) .$$

We used Weierstrass M-test to note uniform convergence in both S and S'

One would like to somehow get

$$S''(x) = -\sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) ,$$

which we recognize from p. 382 as $x - \pi$ for $0 < x \leq \pi$

MUST GO SLOW!!

Harder Example.

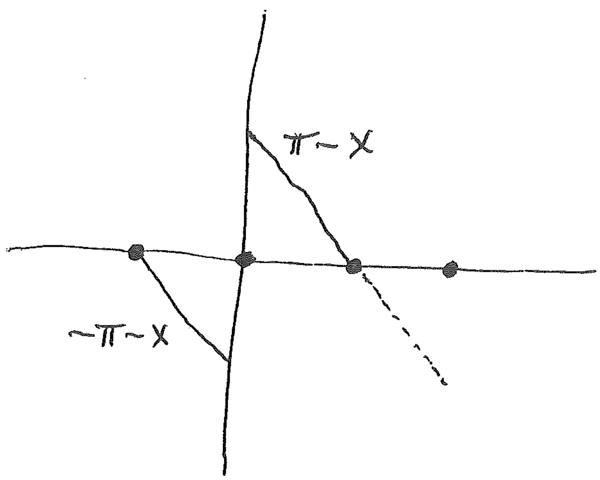
Can we (actually) get $S''(x)$?? One wonders if this might be

$$-\sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) .$$

$M_n = \frac{2}{n}$ is bad.

This series is familiar. Recall:

p. 382



FS was

$$\sum_{n=1}^{\infty} \frac{2}{n} \sin(nx) .$$

p. 9 Feb 15

By the improved Fourier's Theorem, we know that $\sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$ converges uniformly on $[\delta, \pi]$. (Similarly for $[-\pi, -\delta]$.)

$$\left\{ \begin{array}{l} \text{I'll give a } 2^{\text{nd}} \text{ proof} \\ \text{on page 18.} \end{array} \right\}$$

Hence, by the (classical) diff thm, we see that $S''(x) = x - \pi$ on every interval $[\delta, \pi]$.

Thus,

$$S''(x) = x - \pi \quad \text{on} \quad \{0 < x \leq \pi\}.$$

(Similarly with $x + \pi$ for negative x.) ✓✓

Remember that we already knew $S(x)$ is C^1 on $[-\pi, \pi]$. Indeed:

$$S'(x) = \sum_{1}^{\infty} \frac{2}{n^2} \cos(nx) .$$

ok

Without further ado, on $(0, \pi]$, we get:

$$S'(x) = \frac{x^2}{2} - \pi x + C_1$$

$$S(x) = \frac{x^3}{6} - \pi \frac{x^2}{2} + C_1 x + C_2 .$$

But: $S(0) = 0$, $S(\pi) = 0$ by def of S.

Get: $C_2 = 0$, $C_1 = \frac{\pi^2}{3}$. Hence,

$$S(x) = \frac{x^3}{6} - \pi\frac{x^2}{2} + \frac{\pi^2}{3}x \quad \text{on } (0, \pi].$$

Since $S(x)$ is odd (again by def), we finally get:

$$S(x) = \begin{cases} \frac{x^3}{6} - \pi\frac{x^2}{2} + \frac{\pi^2}{3}x, & 0 < x \leq \pi \\ 0, & x = 0 \\ \frac{x^3}{6} + \pi\frac{x^2}{2} + \frac{\pi^2}{3}x, & -\pi \leq x < 0 \end{cases}.$$

Note that:

$$\frac{x^3}{6} - \pi\frac{x^2}{2} + \frac{\pi^2}{3}x = \frac{1}{6}(x^3 - 3\pi x^2 + 2\pi^2 x)$$

$$= \frac{x}{6}(x^2 - 3\pi x + 2\pi^2)$$

$$= \frac{x}{6}(x - 2\pi)(x - \pi).$$ ✓✓

Compare $x(x-1)(x-2)$ on 382 of textbook!!

Table Usage:

$$x(x-1)(x-2) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n^3} \qquad (0<x<1)$$

$$\left\{ \frac{n\pi x}{L} \longrightarrow \frac{n\pi x}{1} \quad \text{OK} \right\}$$

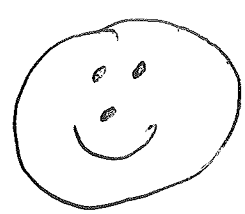
$$y = \pi x \,, \quad x = \frac{y}{\pi} \qquad \text{change of scale}$$

$$\therefore \quad \frac{y}{\pi}\left(\frac{y}{\pi}-1\right)\left(\frac{y}{\pi}-2\right) = 12\sum_{n=1}^{\infty}\frac{\sin ny}{\pi^3 n^3} \qquad (0<y<\pi)$$

$$y(y-\pi)(y-2\pi) = 12\sum_{n=1}^{\infty}\frac{\sin(ny)}{n^3}$$

$$\frac{1}{6}\,y(y-\pi)(y-2\pi) = 2\sum_{n=1}^{\infty}\frac{\sin(ny)}{n^3} \qquad (0<y<\pi)$$

Agree!



B

Given $I = [\alpha, \alpha + 2\pi]$. Given (p. continuous) $f(x)$ on I.

Form $F(x) =$ the (2π) periodic ext of f.

Form $FS(f)$:

$$f \sim \frac{a_0}{2} + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Q: what is $FS(F)$ on $[\omega, \omega + 2\pi]$?

A: the same !! ($\omega =$ arbitrary)

Pf.

$$A_n = \frac{1}{\pi} \int_{\omega}^{\omega + 2\pi} F(x) \cos(nx)\, dx, \quad n \geqq 0$$

$$\left\{ \begin{array}{l} \text{but } F(x)\cos(nx) \text{ is } 2\pi\text{-periodic} \\ \text{can use any "full cycle" in } x \end{array} \right\}$$

$$A_n = \frac{1}{\pi} \int_{\alpha}^{\alpha + 2\pi} F(x) \cos(nx)\, dx$$

{but $F = f$ except finite # of points}

so $A_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx$

$= a_n$, ETC. ∎

The point is that we can apply Fourier's Theorem (or improved version) anew to $F(x)$ on $[\omega, \omega+2\pi]$. This can give a helpful perspective!

Eg, look at p. 2. (graph) "f" on $[-\pi, \pi]$ to F on $[0, 2\pi]$
etc.

Little Theorem (Useful.)

arbitrary

$I = [\alpha, \alpha+2\pi]$, $J = [\omega, \omega+2\pi]$.

$FS(f \text{ on } I) \equiv FS(F \text{ on } J)$

Similarly for $2L$.

(C) Reassuring!

(8)

"Circular Theorem"

Let $I = [\alpha, \alpha + 2\pi]$. Given the series

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

ASSUME that this series conv uniformly on I. Call the sum $S(x)$. Then:

(i) $S(x)$ is continuous on I;

(ii) the FS of function $S(x)$ on I is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

2L

Pf

Recall the classical unif conv thms about continuous fcns and integration.

(i) is thus OK. For (ii), use the integ thm with, eg, $p(x) = \sin(\ell x)$. Get:

Side Remark. better: only need $S(x)$ R-integrable and $|S_N(x)| \leq M$ (Arzela-Osgood)

$$\int_I S(x)\sin(lx)\,dx = \int_I \frac{A_0}{2}\sin(lx)\,dx + \sum_n \int_I \left(A_n \cos nx \sin lx + B_n \sin nx \sin lx\right) dx$$

$$= 0 + 0 + \left(A_l \cdot \underline{0} + B_l \underline{\pi}\right) + 0$$

$$\Downarrow$$

$$B_l = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} S(x)\sin(lx)\,dx\,.$$

Similarly with $S(x)\cos lx$. $\blacksquare$

FORMAT

E.G. (the) **FS** of $\cos 3x + 7\sin(15x)$ **on** **$[-\pi,\pi]$** is $\cos 3x + 7\sin(15x)$.

Compare textbook p.199 (prob 4)

Similarly for "FSS" and "FCS".

(D) Now go to some more complicated theorems.

Book p. 54

THM (differentiation of FS)

Let $f(x)$ be of type (abc) on $I = [\alpha, \alpha + 2\pi]$.
Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

One automatically knows that

$$f'(x) \sim 0 + \sum_{n=1}^{\infty} (n b_n \cos nx - n a_n \sin nx).$$

IN ADDITION: we have equality at any point x_1 in $\{\alpha < x < \alpha + 2\pi\}$ at which (i) f' is continuous and (ii) the one-sided second derivatives $f''_{\pm}(x_1)$ exist.

usually just use $f''(x_1)$

2L

Pf

f is continuous and piecewise C^1 on I. by def

Thus, f' is piecewise continuous on I.

It can only have jumps.

Recall that, due to type (abc), integ by parts gave

$$b_n(f') = \frac{1}{\pi}\int_I f'(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_I \sin(nt)\,d[f(t)]$$

$\boxed{u, v \text{ continuous piecewise } C^1}$

$$= \frac{1}{\pi}\left\{ \sin(nt) f(t) \Big]_{\alpha}^{\alpha+2\pi} - \int_I f(t)\, n\cos(nt)\,dt \right\}$$

$$= \frac{1}{\pi}\left\{ 0 - n\int_I f(t)\cos(nt)\,dt \right\}$$

$$= -(n)\frac{1}{\pi}\int_I f(t)\cos(nt)\,dt$$

$$= -n a_n .$$

Similarly for $a_n(f')$. Note:

$$a_0(f') = \frac{1}{\pi}\int_I f'(t)\,dt = \frac{1}{\pi}\left[f(\alpha+2\pi) - f(\alpha)\right]$$

$$= 0 .$$

We can now apply Fourier's Theorem to $FS(f')$ to get the "IN ADDITION" assertion. $\blacksquare$

Note

One typically addresses $x_1 = \alpha, \alpha + 2\pi$ by shifting matters over to $F(x)$ on a "perturbed" interval $[\omega, \omega + 2\pi]$. Recall p. 7 Little Theorem.

> Also recall that $F(x)$ is CONTINUOUS + p.C^1 (2π) periodic.

NICE!!

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p. 382

$$\pi^2 x - x^3 = 12 \sum_1^\infty \frac{(-1)^{n+1}}{n^3} \sin(nx) \qquad [-\pi, \pi]$$

abc

$$\pi^2 - 3x^2 \sim 12 \sum_1^\infty \frac{(-1)^{n+1}}{n^2} \cos(nx)$$

$f' = \pi^2 - 3x^2$ ✓

$f'' = -6x$ ✓✓

$$\Downarrow$$

$$3x^2 \sim \pi^2 + 12 \sum_1^\infty \frac{(-1)^n}{n^2} \cos(nx)$$

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_1^\infty \frac{(-1)^n}{n^2} \cos(nx)$$

cf. 381

Abel's Lemma

Given real numbers $\{c_1, \dots, c_N\}$ and **positive** numbers $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_N$.

Let $S_k = \sum_{j=1}^{k} c_j$. **Assume** that

$$m \leq S_k \leq \mathfrak{M}, \quad 1 \leq k \leq N.$$

Then:

$$m\underline{\epsilon_1} \leq \sum_{j=1}^{N} \epsilon_j c_j \leq \mathfrak{M}\underline{\epsilon_1}.$$

Pf

$c_k = S_k - S_{k-1}$. Get:

$$\sum_1^N \epsilon_j c_j = \epsilon_1 S_1 + \epsilon_2(S_2 - S_1) + \epsilon_3(S_3 - S_2) + \cdots + \epsilon_N(S_N - S_{N-1})$$

$$= S_1(\epsilon_1 - \epsilon_2) + S_2(\epsilon_2 - \epsilon_3) + \cdots + S_{N-1}(\epsilon_{N-1} - \epsilon_N) + \underline{S_N \epsilon_N}.$$

BUT: $\epsilon_j > 0$, $\epsilon_{j-1} - \epsilon_j > 0$.

Apply elementary inequalities!! Get:

$$\sum_1^N \epsilon_j c_j \lesseqgtr \mathfrak{M}(\epsilon_1 - \epsilon_2) + \mathfrak{M}(\epsilon_2 - \epsilon_3) + \cdots + \mathfrak{M}(\epsilon_{N-1} - \epsilon_N) + \mathfrak{M}\epsilon_N$$

{reverse the telescope!}

$$= \mathfrak{M}\epsilon_1 .$$

How trivial!

Similarly from below. $\blacksquare$

Note that the Lemma immediately adapts to

$$\sum_{j=Q}^{R} \epsilon_j c_j .$$

"ϵ_Q"

Abel's Lemma gives 2 very nice thms about uniform conv. We do #1 today.

THM (Dirichlet's Test for Unif Conv)

Let $I = [a,b]$. Given functions $\{c_j(x)\}_{j=1}^{\infty}$ on I such that

$$\left| \sum_{j=1}^{k} c_j(x) \right| \leq M$$

for all $k \geq 1$ and $x \in I$. Given positive numbers $\{\epsilon_j\}_{j=1}^{\infty}$ such that

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots, \quad \lim_{j\to\infty} \epsilon_j = 0.$$

The infinite series

$$\sum_{j=1}^{\infty} \epsilon_j c_j(x)$$

is then UNIFORMLY conv on I.

Pf

Let $S_k(x) = \sum_{j=1}^{k} c_j(x)$. Review the proof of

Abel's Lemma. Notice that

$$(\ast) \qquad \left| \sum_{j=\ell}^{\ell+R} c_j(x) \right| \leq \underline{\underline{2\mathcal{M}}} \qquad \left\{ \begin{matrix} \text{all } R \geq 0,\ \ell \geq 1 \\ x \in I \end{matrix} \right\}.$$

For $x \in I$, we have p. 13

$$\sum_{j=1}^{N} \epsilon_j c_j(x) = \sum_{j=1}^{N-1} s_j(x)(\epsilon_j - \epsilon_{j+1}) + s_N(x)\epsilon_N .$$

We seek to let $N \to \infty$. Since

$$\sum_{j=1}^{\infty} \mathcal{M}(\epsilon_j - \epsilon_{j+1}) = \mathcal{M}\epsilon_1 < \infty ,$$

series $\sum_{j=1}^{\infty} s_j(x)(\epsilon_j - \epsilon_{j+1})$ is <u>absolutely</u> conv;
in addition, $\lim_{N\to\infty} s_N(x)\epsilon_N = 0$ for each $x \in I$.
It follows that the series $\boxed{\text{use } N \to \infty}$

$$\mathcal{S}(x) \equiv \sum_{j=1}^{\infty} \epsilon_j c_j(x) \quad \text{converges,}$$

for each $x \in I$. Let $\mathcal{S}_N(x) = \sum_{j=1}^{N} \epsilon_j c_j(x)$.
Notice that <u>Abel's Lemma</u> applies to

$$\mathcal{S}_{N+r}(x) - \mathcal{S}_N(x) = \sum_{j=N+1}^{N+r} \epsilon_j c_j(x) \qquad \underline{\underline{r \geq 1}}$$

using $(\ast)$. $\leftarrow$ $\boxed{\text{This is a trick.}}$

As such:

$$-(2M)\,\epsilon_{N+1} \leq s_{N+r}(x) - s_N(x) \leq (2M)\,\epsilon_{N+1}$$

for each $x \in I$, $N \geq 1$, $r \geq 1$. We now take $r \to \infty$ to get

$$|s(x) - s_N(x)| \leq (2M)\,\epsilon_{N+1},$$

for each $x \in I$. Since $\epsilon_{N+1} \to 0$ as $N \to \infty$, this is enough to prove the UNIF conv of series $s(x)$ on I. ▥

NICE! SLICK!!

In the above, nothing excludes taking $a = b$ and $I = \{a\}$ = one point. In that case, Dirichlet's Test speaks about ordinary convergence of a single infinite series

$$\sum_{j=1}^{\infty} \epsilon_j \, c_j \, .$$

no x

F

THEOREM. Let

$$\epsilon_1 > \epsilon_2 > \cdots \; ; \quad \lim_{j \to \infty} \epsilon_j = 0 .$$

Then,

$$\sum_{n=1}^{\infty} \epsilon_n \cos(nx) \quad \text{and} \quad \sum_{n=1}^{\infty} \epsilon_n \sin(nx)$$

are UNIF CONV on each $[\delta, 2\pi - \delta]$.

$$\boxed{\sum_{n=Q}^{\infty} \text{ ok too.}}$$

E.G.

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin(nx) , \qquad \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)$$

$$\sum_{n=3}^{\infty} \frac{1}{\ln(n)} \sin(nx) , \qquad \sum_{n=3}^{\infty} \frac{1}{\ln(n)} \cos(nx)$$

Impressive

Pf

Given δ. Need to get:

$$\left|\cos t + \cdots + \cos(Nt)\right| \leqq M$$

$$\left|\sin t + \cdots + \sin(Nt)\right| \leqq M$$

on $\delta \leqq t \leqq 2\pi - \delta$ for <u>some</u> M. $(N \geqq 1)$

$$1 + e^{it} + \cdots + e^{Nit} = \frac{e^{(N+1)it} - 1}{e^{it} - 1}$$

$$= \frac{e^{(N+\frac{1}{2})it} - e^{-it/2}}{e^{it/2} - e^{-it/2}}$$

$$= \frac{e^{(N+\frac{1}{2})it} - e^{-i\frac{t}{2}}}{2i\sin\left(\frac{t}{2}\right)}$$

$$= \frac{-i}{2\sin\left(\frac{t}{2}\right)}\left[\cdots\right]$$

$$\Downarrow$$

$$= \frac{\sin(N+\frac{1}{2})t + \sin(\frac{t}{2})}{2\sin(\frac{t}{2})}$$

$$-i\left[\frac{\cos(N+\frac{1}{2})t - \cos(\frac{t}{2})}{2\sin(\frac{t}{2})}\right]$$

$$\Downarrow$$

so,

$$\cos t + \cdots + \cos(Nt) = \frac{\sin(N+\frac{1}{2})t}{2\sin(\frac{t}{2})} - \frac{1}{2} \quad \checkmark$$

Dirichlet Kernel

$$\sin t + \cdots + \sin(Nt) = \frac{\cos(\frac{t}{2}) - \cos(N+\frac{1}{2})t}{2\sin(\frac{t}{2})}$$

new

$$\delta \leq t \leq 2\pi - \delta$$

$$\Rightarrow \frac{\delta}{2} \leq \frac{t}{2} \leq \pi - \frac{\delta}{2}$$

$$\Rightarrow \frac{1}{\sin(\frac{t}{2})} \leq \frac{1}{\sin(\frac{\delta}{2})}$$

$$\Downarrow$$

$$\boxed{M = \frac{2}{2\sin(\frac{\delta}{2})}}$$

works

To finish up, we just apply Dirichlet's Test for Unif Conv on $[\delta, 2\pi - \delta]$. Page 15. $\blacksquare$

Baby L'Hôpital's Rule $\Rightarrow$

$$\lim_{v \to 0^+} \left(\frac{1}{\sin v} - \frac{1}{v} \right) = 0 .$$

We want to improve this a little.

Recall our proof of Fourier's Thm. We considered

$$\mathcal{C}(v) = \left\{ \begin{array}{ll} \dfrac{v}{\sin v} , & 0 < |v| < \pi \\ 1 , & v = 0 \end{array} \right\} .$$

Good function! Continuous; even; ≥ 1.

It's also C^∞. Near $v = 0$, just invert

$$\frac{\sin v}{v} = 1 - \frac{v^2}{3!} + \frac{v^4}{5!} \pm \cdots$$

$$\Rightarrow \quad \mathcal{C}(v) = 1 + \frac{1}{6} v^2 + a_4 v^4 + a_6 v^6 + \cdots$$

$$= \text{nice power series}$$

$\Rightarrow$ $\boxed{\text{OK}}$

For $v \neq 0$, notice that:

$$\frac{1}{\sin v} - \frac{1}{v} = \frac{\mathcal{C}(v)}{v} - \frac{1}{v}$$

$$= \frac{\mathcal{C}(v) - 1}{v} .$$

For small $|v|$, we get:

$$\frac{1}{\sin v} - \frac{1}{v} = \frac{1}{6} v + a_4 v^3 + a_6 v^5 + \cdots$$

$=$ nice power series.

This <u>refines</u> the L'Hôpital result!

Let

$$\mathcal{D}(v) = \begin{cases} \frac{1}{\sin v} - \frac{1}{v}, & 0 < |v| < \pi \\ 0, & v = 0 \end{cases} .$$

Have $\mathcal{D}(v)$ continuous, odd, C^∞ on $(-\pi, \pi)$.