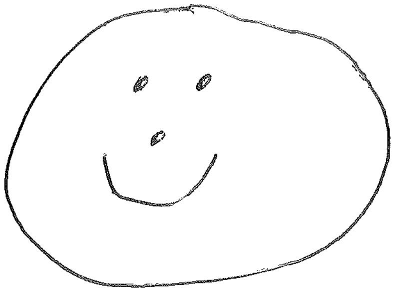
Agenda

$\boxed{1}$ Finish topic ⑥ from Wed.

$\boxed{2}$ RMS-style convergence, L.I.M.

$\boxed{3}$ Gibbs phenomenon



Final 2 pages of Ⓖ

$$D_N(t) = \frac{1}{2} + \cos t + \cdots + \cos(Nt)$$

$$D_N(t) = \frac{\sin\left[(N+\frac{1}{2})t\right]}{2\sin(\frac{t}{2})} \qquad 0 < |t| < 2\pi$$

$$\int_0^{\pi} D_N(t)\,dt = \frac{\pi}{2} \qquad N \geqq 0$$

$$\int_0^{\pi} \frac{\sin(N+\frac{1}{2})t}{2\sin(\frac{t}{2})}\,dt = \frac{\pi}{2}$$

$$\left\{ t = 2v\,,\quad v = \frac{t}{2} \right\}$$

$$\int_0^{\pi/2} \frac{\sin(2N+1)v}{\sin v}\,dv = \frac{\pi}{2} \qquad (N \geqq 0)$$

$$\int_0^{\pi/2} \sin(2N+1)v \cdot \left\{ \frac{1}{v} + \underline{\mathcal{D}(v)} \right\} dv = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}(2N+1)} \frac{\sin w}{w}\,dw + \int_0^{\pi/2} \sin(2N+1)v \cdot \underline{\mathcal{D}(v)}\,dv = \frac{\pi}{2}$$

$$\boxed{\mathcal{D}(v) = \left\{ \begin{matrix} \frac{1}{\sin v} - \frac{1}{v} \\ 0 \end{matrix} \right\} \quad -\pi < v < \pi} \longleftarrow \text{p. ㉓ previous Lec}$$

$$\lim_{N\to\infty} \int_0^{\pi/2} \overset{C^\infty}{\underline{\underline{\mathcal{D}(v)}}} \cdot \sin(2N+1)v \, dv = 0$$

by R–L Lemma! (or parts) !!

hence,

$$\lim_{N\to\infty} \int_0^{\pi(N+\frac{1}{2})} \frac{\sin w}{w} \, dw = \frac{\pi}{2} \quad \surd\surd$$

but: $\int_0^\infty \frac{\sin w}{w} \, dw$ converges as an improper integral (book p. 164, by use of integ by parts)

$$\therefore \quad \boxed{\int_0^\infty \frac{\sin w}{w} \, dw = \frac{\pi}{2}}$$

FAMOUS INTEGRAL

$\boxed{\text{RECALL}}$

Root Mean Square Distance between graphs $y = f(x)$, $y = g(x)$ for $a \leq x \leq b$:

$$\sqrt{\frac{1}{b-a}\int_a^b \left(f(x)-g(x)\right)^2 dx} \; .$$

As I said earlier, people frequently ignore the $\frac{1}{b-a}$. I.E., they just look at

$$\|f-g\| .$$

There is a type of convergence that's associated with $\| \ \|$.

We say $f_n(x) \rightarrow g(x)$ in the mean on $\underset{[a,b]}{\underset{\uparrow}{I}}$ as $n \to \infty$ when

$$\lim_{n\to\infty} \|f_n - g\| = 0 .$$

Equivalently,

$$\lim_{n\to\infty} \int_a^b (f_n(x)-g(x))^2\,dx = 0$$

OR

$$\|f_n - g\| < \varepsilon \quad \text{anytime} \quad n \geqq \underline{N_\varepsilon}$$

(for each $\varepsilon > 0$).

One sometimes writes chap 7 p.203 !

$$\underset{n\to\infty}{\text{L.I.M.}}\ f_n(x) = g(x) .$$

Parseval's Thm on $[\alpha, \alpha+2\pi]$ for FS says that

$$\underset{n\to\infty}{\text{L.I.M.}}\ S_n(x) = f(x),$$

for every piecewise continuous f.

Unfortunate Fact.

$I = [a, b]$. *Unless* matters are restricted somehow, there is **NOT** a *simple* relationship between the statements

$$\lim_{n \to \infty} f_n(x) = g(x) \quad \text{on} \quad I$$

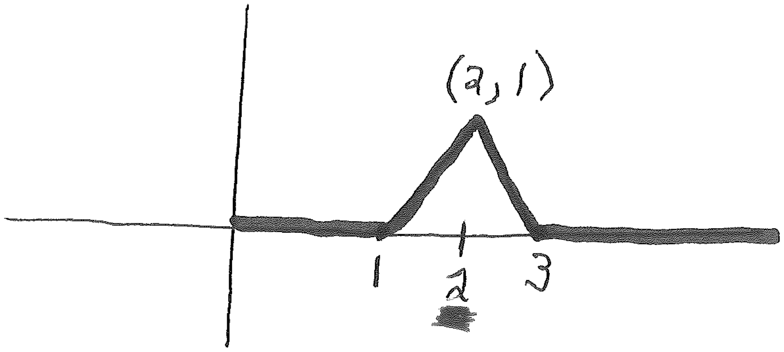
and

$$\underset{n \to \infty}{\text{L.I.M.}}\, f_n(x) = g(x) \quad \text{on} \quad I.$$

The book points this out, too, around pp. 203, 204, 209 (prob 5) in chapter 7.

It's worthwhile to show some slightly different examples!

Let $L(x)$ be the piecewise linear isosceles triangle function



$\{0 \leq x < \infty\}$

Obviously,

$$\int_0^\infty L(x)\,dx = 1 .$$

↑ continuous

C^k ?

Example 1

Take $I = [0, 5]$, say. Let

$$f_n(x) = n^{\frac{1}{2}} \sqrt{L(nx)} , \quad n \geq 1 .$$

Let $g(x) = 0$. Then,

$$\lim_{n \to \infty} f_n(x) = g(x) = 0 \quad \text{on } I ,$$

but

$$\|f_n - g\| \not\to 0 .$$

Verification

Know that $L(t) > 0 \iff 1 < t < 3$.

Note that $f_n(0) = 0$; also $f_n(x) > 0$

$\iff \frac{1}{n} < x < \frac{3}{n}$.

$\boxed{\max f_n = \sqrt{n}}$

By the latter, we get:

$$\lim_{n\to\infty} f_n(x_0) = 0$$ for **each** $x_0 \in (0, 5]$.

$\boxed{\text{WAIT til } n \text{ is big}}$

Thus, $\lim_{n\to\infty} f_n(x) = 0 = g(x)$.

But:

$$\|f_n - g\|^2 = \int_0^5 f_n(x)^2\,dx = \int_0^5 n\,L(nx)\,dx$$

$$= \int_0^{5n} L(y)\,dy = \underline{\underline{1}}$$ (since $n \gtrsim 1$).

So, $\|f_n - g\| \not\to 0$. ■

Example 2

$I = [0,5]$. Keep $n \geqq 10$, say.

Declare $L(x) \equiv 0$ for $x < 0$.

Let $g(x) = 0$. Let $1 < c < 4$.

Let
$$f_n(x) = \sqrt{L[2 + n(x-c)]} \, .$$

Then:
$$\underset{n \to \infty}{\text{L.I.M.}} \; f_n(x) = g(x) = 0 \, ,$$

but
$$\lim_{n \to \infty} f_n(x) = \left\{ \begin{array}{ll} 0, & x \neq c \\ 1, & x = c \end{array} \right\} \neq g \, .$$

CAN TAKE A SUM OF A "FEW" OF THESE (different c's). GET EXAMPLES (where) $\lim_{n \to \infty} f_n(x)$ different from 0 at $\underline{\underline{10^9}}$ points.

(etc)

or doesn't exist
via slight modification

Pf of Example

$$\overset{f_n(x)}{\sqrt{L[2+n(x-c)]}} > 0 \iff -\frac{1}{n} < x-c < \frac{1}{n}$$

$$\iff c-\frac{1}{n} < x < c+\frac{1}{n} \, .$$

For <u>each</u> x_0 in $[0,5]$, $x_0 \neq c$, if you <u>wait</u> until n is big enough, $f_n(x_0) = 0$. For $x_0 = c$, have $f_n(x_0) = f_n(c) = 1$ all n.

So,

$$\lim_{n\to\infty} f_n(x) = \begin{Bmatrix} 0, & x \neq c \\ 1, & x = c \end{Bmatrix} . \qquad \boxed{OK}$$

But, NOTE,

$$\|f_n - g\|^2 = \int_0^5 f_n(x)^2 \, dx \qquad \{g \equiv 0\}$$

$$= \int_{c-\frac{1}{n}}^{c+\frac{1}{n}} L[2+n(x-c)] \, dx$$

$$= \int_1^3 L[y] \frac{dy}{n} = \frac{1}{n} \to 0 .$$

So, $\operatorname{l.i.m.}_{n\to\infty} f_n(x) = 0$. $\boxed{OK}$ ∎

Gibbs Phenomenon

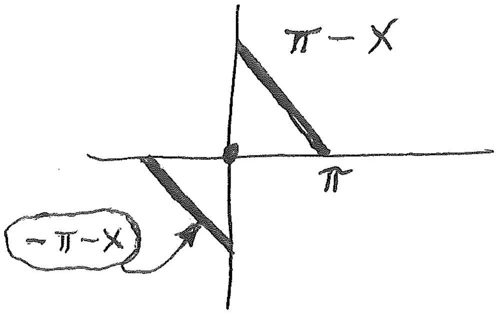
~ book p. 52

similar example!

"Gibbs Overshoot" at jump discontinuities

Recall—

$$S(x) = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx) \quad , \quad -\pi \leqq x \leqq \pi$$



Convergence cannot be uniform on $[-\delta, \delta]$ (why?)

For N giant, we wonder about the shape of

$$S_N(x) \equiv \sum_{n=1}^{N} \frac{2}{n}\sin(nx) \, .$$

Can we figure this out??

$S_N(0) = 0$; $S_N(x)$ odd.

$$S_N'(x) = \sum_{n=1}^{N} 2\cos(nx) \qquad \boxed{S_N'(0) = 2N}$$

but

$$\text{RHS} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} - 1 \qquad \begin{matrix} p.① \\ \{0 < x < 2\pi\} \end{matrix}$$

$$= \frac{\sin(N+\frac{1}{2})x - \sin\frac{x}{2}}{\sin\frac{x}{2}}$$

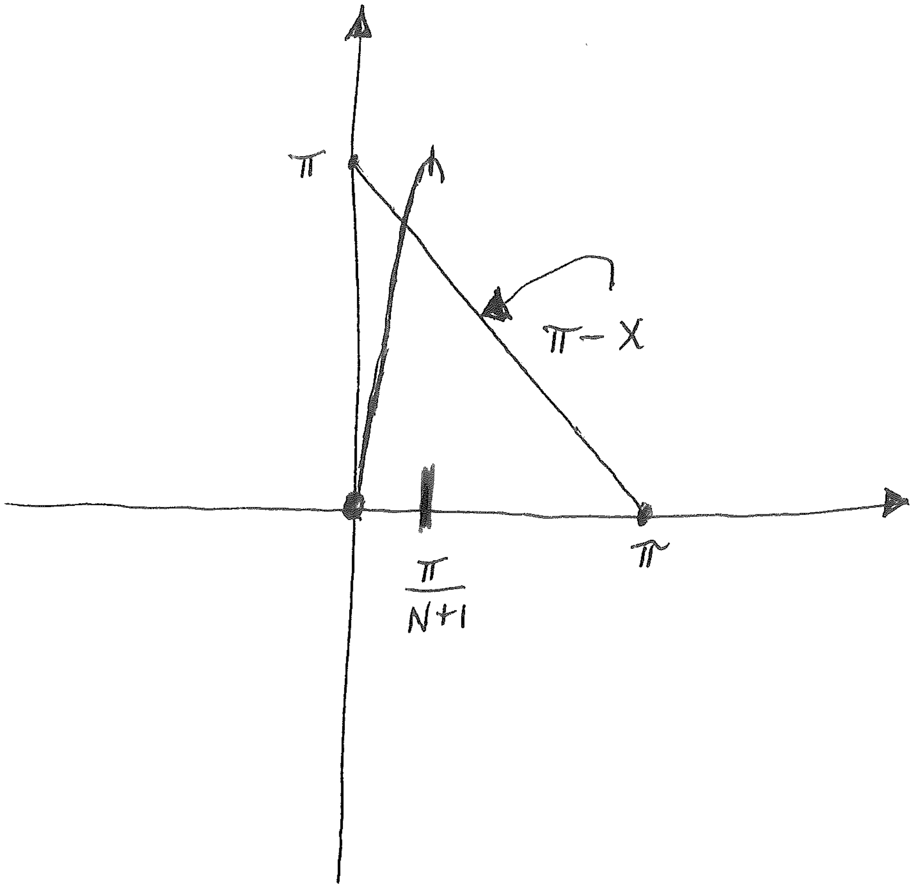
$$\left\{ \begin{aligned} & A = \left(\frac{N}{2}+\frac{1}{2}\right)x, \quad B = \frac{N}{2}x \\ & A+B = \left(N+\frac{1}{2}\right)x \\ & A-B = \frac{x}{2} \quad \leftarrow \text{easy} \\ & \sin(A+B) - \sin(A-B) = 2\cos A \sin B \end{aligned} \right\}$$

Alternate Form:

$$S_N'(x) = \frac{2\cos\frac{N+1}{2}x \cdot \sin\frac{N}{2}x}{\sin\frac{x}{2}}$$

Clearly $S_N'(x) > 0$ for $0 < x < \frac{\pi}{N+1}$. ← plug it in!

$$\boxed{\frac{\pi}{N+1} = \text{local max}}$$



Very steep!

How high is the local max?

Know:

$$S_N'(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} - 1 \quad , \quad 0 < x < 2\pi$$

so

$$S_N(x) = \int_0^x \left[\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}} - 1 \right] dt \quad .$$

Keep $0 < x \lessapprox \pi$ for safety!!

but,

$$\frac{1}{\sin y} = \frac{1}{y} + \mathcal{D}(y), \quad 0 \leq y \leq \frac{\pi}{2}$$

C^∞ + bdd

power series
odd powers
of y

so,

$$S_N(x) = -x + \int_0^x \sin(N+\tfrac{1}{2})t \cdot \left\{ \frac{2}{t} + \mathcal{D}\left(\frac{t}{2}\right) \right\} dt$$

$$S_N(x) = -x + 2\int_0^x \frac{\sin(N+\frac{1}{2})t}{t}\,dt$$

$$+ \int_0^x \mathcal{D}\left(\frac{t}{2}\right) \sin(N+\tfrac{1}{2})t\,dt$$

$$\boxed{\begin{aligned} S_N(x) &= -x + 2\int_0^{(N+\frac{1}{2})x} \frac{\sin u}{u}\,du \\ &\quad + \int_0^x \overset{C^\infty}{\mathcal{D}\left(\frac{t}{2}\right)} \sin(N+\tfrac{1}{2})t\,dt \end{aligned}}$$

E.G., Fix (any) x in $(0,\pi]$. Let $N \to \infty$ and use R-L lemma. Get:

$$S(x) = -x + 2\int_0^\infty \frac{\sin u}{u}\,du + \underline{\underline{0}} \; .$$

Hence, we *must* have { if we didn't know it already }

$$\Bigg\| \quad \frac{\pi}{2} = \int_0^\infty \frac{\sin u}{u}\,du \; . \quad \Bigg\|$$

Though I omit the proof, I note that a *simple* use of **integration by parts** in the earlier big black box shows that $S_N(x) \to \pi - x$ **UNIFORMLY** on $[\delta, \pi]$ for any $\delta > 0$. Ie,

proof #3

$$\left| S_N(x) - (\pi - x) \right| < \varepsilon$$

for all $N \geq$ some N_ε .

Play with $\boxed{\mathcal{D}\text{-integral}}$ AND $\boxed{\int_T^\infty \frac{\sin w}{w}\,dw}$!!!

Finally, we arrive at

Last step: put $\lambda = \frac{\pi}{N+1}$.

Need $S_N(\lambda)$ for N large! p. ⑫

By big black box, for each N, we get p. ⑬

$$S_N(\lambda) \approx -\lambda + 2\int_0^{(N+\frac{1}{2})\lambda} \frac{\sin u}{u}\,du$$

$$+ \int_0^{\lambda} \mathcal{D}\left(\frac{t}{2}\right) \sin\left(N+\frac{1}{2}\right)t\,dt.$$

Note that:

$$|-\lambda| \approx \frac{\pi}{N+1} \quad ;$$

$$\left| \int_0^{\lambda} \mathcal{D}\left(\frac{t}{2}\right) \sin\left[\left(N+\frac{1}{2}\right)t\right] dt \right| \lesssim \int_0^{\lambda} \left| \overbrace{\mathcal{D}\left(\frac{t}{2}\right)}^{\boxed{C^\infty + \text{bdd}}} \right| dt$$

$$\lesssim \frac{C_1}{N+1} \quad ;$$

$$(N+\tfrac{1}{2})\lambda = (N+1)\lambda - \tfrac{1}{2}\lambda \quad \text{trivially}$$

$$= \underline{\pi} - \frac{\pi}{2(N+1)} \quad .$$

But, now,

$$2\int_0^{(N+\frac{1}{2})\lambda} = 2\int_0^{\underline{\pi}} - 2\int_{\pi-\frac{\pi}{2(N+1)}}^{\pi} \quad ;$$

$$\left| 2\int_{\pi-\frac{\pi}{2(N+1)}}^{\pi} \frac{\sin u}{u}\,du \right| \leqq 2\int_{\pi-\frac{\pi}{2(N+1)}}^{\pi} 1\,du$$

$$\leqq \frac{\pi}{N+1} \quad .$$

So, get

$$S_N(\lambda) = \underline{-\lambda} + 2\int_0^{\underline{\pi}} \frac{\sin u}{u}\,du + \left[\begin{matrix}\text{abs value}\\ \leqq \frac{\pi}{N+1}\end{matrix}\right]$$

$$+ \left[\begin{matrix}\text{abs value}\\ \leqq \frac{C_1}{N+1}\end{matrix}\right]$$

$$= \underline{2\int_0^{\underline{\pi}} \frac{\sin u}{u}\,du} + \left[\begin{matrix}\text{abs value}\\ \leqq \frac{\pi+\pi+C_1}{N+1}\end{matrix}\right]$$

$$\left\{\begin{matrix}\underline{\text{numerical integration}}\ \text{gives}\\ \int_0^{\pi} \frac{\sin t}{t}\,dt = \pi(.589489^{+})\end{matrix}\right\} \quad ;$$

so,

$$\boxed{S_N(\lambda) = \left[\text{term of abs. value} \lessapprox \frac{c_2}{N+1}\right] + 2\pi(.589489^{+})}\,.$$

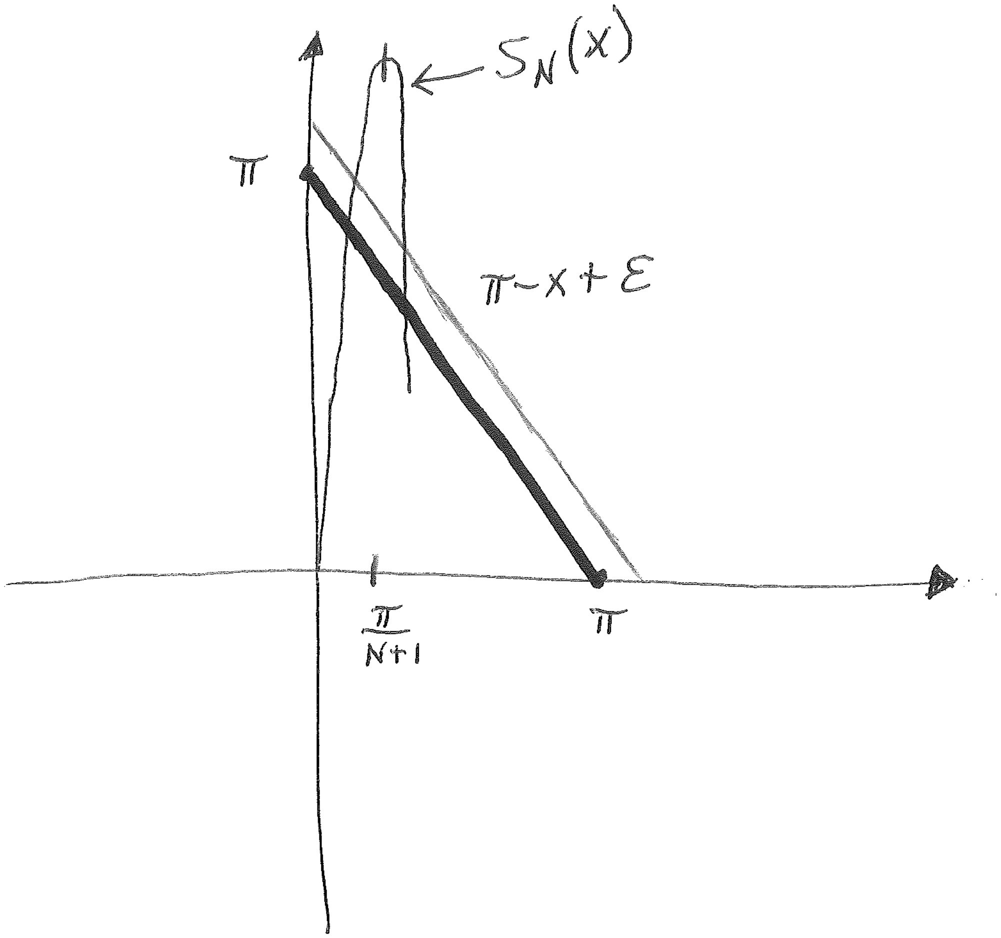
So, for N large, we get an <u>overshoot</u> <u>vs.</u> $2\pi(.500000)$ with ratio

$$1.178978^{+}\,.$$

There will be a corresponding ratio in the size of the $\pm\frac{\pi}{N+1}$ <u>"jump"</u>.

$$\boxed{\text{This extra} \sim 18\% \text{ is seen in } \underline{\text{any}} \text{ FS situation where}^{\text{(the)}}\text{ piecewise } C^1 \text{ function } f(x) \text{ has a jump discontinuity.}}$$

(Book p. 53: $\dfrac{\sigma - \frac{\pi}{2}}{\pi/2} \cong .178$)



$$S_N(x) \not\longrightarrow \pi - x$$

uniformly on $0 < x \leq \pi$

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note the overshoot near $x = 0$