Last time, I stated and gave a careful proof (step-by-step) of Fourier's Theorem.

If you "revisit" my proof, you can see that we actually have a better theorem.

Better Version of Fourier's Theorem

Given piecewise continuous $h(x)$ on $[a, a+2L]$. Form 2L-periodic extension $H(x)$ of $h(x)$. Form $FS(h)$. Assume that $H'_+(x_0)$ and $H'_-(x_0)$ exist as finite numbers at a point $x_0 \in \mathbb{R}$. Then: $FS(h)$ converges at point x_0, and its value is

$$\frac{H(x_0+0) + H(x_0-0)}{2}.$$

point by point

Recall the "reason"!! You just had to make sure that

$$\frac{H(x_0+v)-H(x_0+0)}{v}\,c(v)$$

and

$$\frac{H(x_0+v)-H(x_0-0)}{v}\,c(v)$$

were piecewise continuous on $[0,\pi]$ and $[-\pi,0]$, respectively. The only possible snag was at $v=0$; i.e., $v\to 0^+$ and $v\to 0^-$.

QED

If $H'_+(x_0)$, $H'_-(x_0)$ exist, we get good one-sided limits at $v=0$.

We stress here that

$$FS(h) = \underline{\underline{H(x_0)}}$$

at any point $x_0 \in \mathbb{R}$ at which

(i) H is <u>continuous</u> has no jump

<u>and</u>

(ii) $H'_{+}(x_0)$ and $H'_{-}(x_0)$ exist.

Please note:

$$\boxed{\begin{array}{l}\text{(ii) is AUTOMATIC anytime} \\ h(x) \text{ is } \underline{\underline{\text{piecewise } C^1}} \text{ on } [a, a+2L]\end{array}}$$

Notice that, in (both) basic Fourier's Thm and better Fourier's Theorem, we had to sum our series with the cosine - sine grouping

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

Reason: we needed to write partial sum $S_N(x_0)$ as a nice integral!

It would be NICE if we could always do the A_n and B_n series separately!

Given f on $[a, a+2L]$. Get:

$$f \sim \frac{1}{2}A_0 + \sum_1^\infty \left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right).$$

Fact (Useful)

If $f(x)$ is piecewise C^1 on $[a, a+2L]$,

the series

$$\sum_1^\infty A_n \cos\frac{n\pi x_0}{L}, \quad \sum_1^\infty B_n \sin\frac{n\pi x_0}{L}$$

are automatically convergent at each $x_0 \in \mathbb{R}$.

NO WORRIES.

Skip the full proof. Just show the reasoning for $L = \pi$ and interval

$$[a, a+2L] = [-\pi, \pi].$$

Let $g(x) = \dfrac{f(x)+f(-x)}{2}$, $h(x) = \dfrac{f(x)-f(-x)}{2}$.

Note:

g is even, h is odd. $\checkmark\checkmark$

$$\boxed{f = g+h}$$

Consider g. Know:

$$B_n(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{g(x)\sin(nx)}_{\text{ODD}}\,dx = \underline{0};$$

$$A_n(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x)+f(-x)}{2}\cos(nx)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$+ \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(-x)\cos(nx)\,dx}_{x=-y}$$

$$\frac{1}{2\pi}\int_{\pi}^{-\pi} f(y)\cos(ny)(-1)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cos(ny)\,dy$$

So:

p. 5

$$A_n(g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \underline{\underline{A_n}} \;;$$

$$B_n(g) = \underline{\underline{0}} \;;$$

$$FS(g) \equiv \frac{1}{2}A_0 + \sum_{1}^{\infty}\left(A_n \cos nx + 0\right).$$

Aha!! Similarly,

$$FS(h) \equiv 0 + \sum_{1}^{\infty}\left(0 + B_n \sin nx\right).$$

But, $g(x)$ and $h(x)$ are <u>obviously</u> <u>piecewise</u> <u>C^1</u> on $[-\pi, \pi]$ since they are linear combinations of two piecewise C^1 functions. So, we can just apply (basic) Fourier's Thm to g and h!

Slick Trick. $\Longrightarrow$ <u>QED</u>

Makes perfectly good sense.

$$f = g + h \text{, as above}$$

$$FS(f) = FS(g) + FS(h)$$

A_n part

B_n part

Great!

We now want to figure out the analog of basic Fourier's Theorem for FCS and FSS.

That is: for a given $f(x)$ on $[0, L]$ which is piecewise C^1.

main case

KEY THING: we know that

$$(*) \quad FS(f_{even}) \equiv FCS(f) \, ;$$

$$(**) \quad FS(f_{odd}) \equiv FSS(f) \, .$$

This is the trick!

Go VERY slow!

FCS first

(*) $FCS(f) \equiv FS(f_{even})$

$f(x)$ on $[0, L]$

f_{even} on $[-L, L]$

N. B.

$\boxed{f_{even}(L) = f_{even}(-L) = f(L)}$

$F_E(x) =$ the $2L$-periodic extension of f_{even}

stands for "even"

need $F_E(-L) = F_E(L)$!

Fudge-able

can use $f(L)$; or 0 ← lazy

Easy to check: F_E even.

F_E often called the even $2L$-periodic extension of $f(x)$

Apply (*) and Fourier's Thm for FS!

Therefore,

FCS at x_0 is $\dfrac{F_E(x_0+0) + F_E(x_0-0)}{2}$.

(10 $\frac{1}{2}$)

$f(x)$ on $[0, \pi]$ $\quad$ $L = \pi$

$f_{even}(x)$ on $[-\pi, \pi]$

Arbitrary x_0.

$$-\underline{\underline{\pi}} + 2\pi k < x_0 < \underline{\underline{\pi}} + 2\pi k \quad \text{interval } \# k$$

$$\Rightarrow \quad F_E(x_0) = \underline{\underline{f_{even}}}(x_0 - 2\pi k) \quad \circledast$$

Flip it!

$$-\pi - 2\pi k < -x_0 < \pi - 2\pi k$$

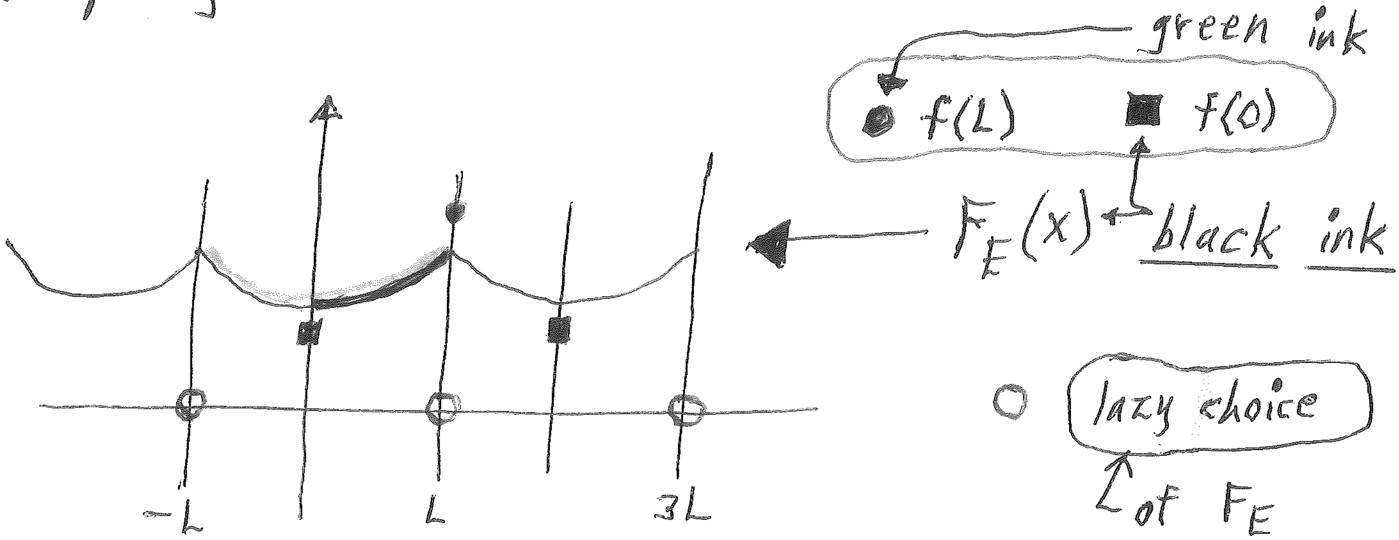
$$-\pi + 2\pi(-k) < -x_0 < \pi + 2\pi(-k) \quad \text{interval } \#(-k)$$

$$\Rightarrow \quad F_E(-x_0) = \underline{\underline{f_{even}}}[-x_0 + 2\pi k] \quad \circledast$$

but f_{even} is even on $[-\pi, \pi]$!!

$$\therefore \quad F_E(-x_0) = F_E(x_0) \qquad \text{OK}$$

In practice, very important to simplify this (p. 10) when $0 \leqq x_0 \leqq L$.



$F_E(x) = f(x)$ for $0 < x < L$ by def

$0 < x_0 < L$ get $\frac{1}{2}\left[f(x_0+0) + f(x_0-0)\right]$

$x_0 = 0$ $\quad F_E(0+) = f(0+)$

$F_E(0-) = f_{even}(0-) = f(0+)$

Average is $f(0+)$.

$x_0 = L$ $\quad F_E(L-) = f(L-)$,

$F_E(L+0) = F_E[(-L)+0]$ $\boxed{2L\text{-periodic}}$

$= f_{even}[(-L)+0] = f(L-)$

Average is $f(L-)$.

$\boxed{\text{SAVE THESE.}}$

FSS next

$(**)\quad FSS(f) \equiv FS(\underline{f_{odd}})$

$f(x)$ on $[0, L]$

f_{odd} on $[-L, L]$

$$\boxed{\begin{aligned} f_{odd}(0) &= 0 \\ f_{odd}(L) &= f(L) \\ f_{odd}(-L) &= -f(L) \end{aligned}}$$

N.B.

odd fcns must be 0 at origin !!

$F_{\theta}(x) =$ the $\underline{2L}$-periodic extension $\underline{\text{of}}$ $\underline{f_{odd}}$

stands for "odd"

need $F_{\theta}(-L) = F_{\theta}(L)$ <periodicity>

Fudge-able values

but, CAUTION, $F_{\theta}(x)$ __can__ __be__ __odd__

__only__ when

$$F_{\theta}(-L) = -F_{\theta}(L) \quad !!$$

So, putting

$$F_{\theta}(-L) = F_{\theta}(L) = \underline{0}$$

lazy is best

is best. Vital!

$\Rightarrow$ Easy to check: F_{θ} is odd.

F_θ is often called the odd $2L$-periodic extension of $f(x)$

Apply (**) and Fourier's Thm for FS.

Therefore,

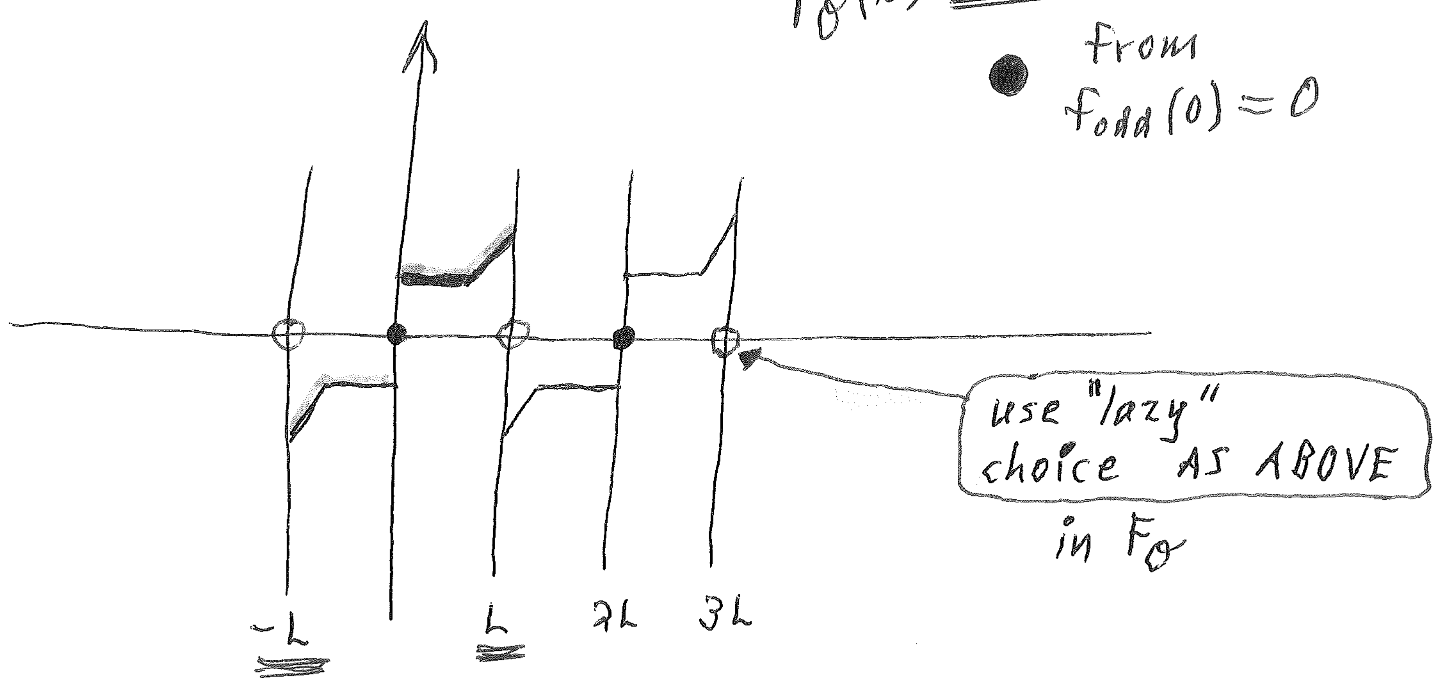
$$\text{FSS at } x_0 \text{ is } \frac{F_\theta(x_0+0)+F_\theta(x_0-0)}{2}.$$

Here, $-\infty < x_0 < \infty$.

In practice, very important to simplify this when $0 \leqq x_0 \leqq L$.

$F_\theta(x)$ in black ink

● from $f_{odd}(0)=0$



$F_{\theta}(x) = f(x)$ for $0 < x < L$

$0 < x_0 < L$ get $\frac{1}{2}[f(x_0+0) + f(x_0-0)]$ ||

$x_0 = 0$ $\quad F_{\theta}(0+) = f(0+)$

$F_{\theta}(0-) = f_{odd}(0-) = -f(0+)$

Average is 0. ||

$x_0 = L$ $\quad F_{\theta}(L-) = f(L-)$

$F_{\theta}(L+0) = F_{\theta}((-L)+0)$ $\boxed{2L\text{-periodic}}$

$= f_{odd}[(-L)+0]$

$= -f[L-0]$

$= -f(L-)$

Average is 0. ||

SAVE THESE

Theorem (for FCS and FSS)

Let $f(x)$ be piecewise C^1 on $[0, L]$.
Form FCS and FSS of f as usual; i.e.,

$$f \sim \frac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{L}$$

$$f \sim \sum_1^\infty b_n \sin\frac{n\pi x}{L} \,.$$

We then have:

$$\mathrm{FCS}(f)[\text{at } x] = \begin{cases} f(0+), & x = 0 \\ \frac{1}{2}[f(x+0) + f(x-0)], & 0 < x < L \\ f(L-), & x = L \end{cases}$$

$$\mathrm{FSS}(f)[\text{at } x] = \begin{cases} 0, & x = 0 \\ \frac{1}{2}[f(x+0) + f(x-0)], & 0 < x < L \\ 0, & x = L \end{cases} .$$

In particular, if x_0 is a point of continuity of f inside $\{0 < x < L\}$, the FCS and FSS will each sum to $f(x_0)$.

When asked to find graphs of a FCS or FSS <u>outside</u> the given interval $[0, L]$, it is most common to <u>go back to</u> FS(f_{even}) or FS(f_{odd}) and then use Fourier's Theorem <u>for</u> <u>FS</u>.

You will need to form the <u>$2L$</u>-periodic extension of your f_{even}, f_{odd}.

Since only one-sided limits are essential, one can proceed [in this] using ANY convenient choice of ordinates at $x = \pm L$.

Conclude by applying

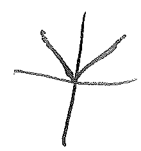
$$\frac{1}{2}\left[H(x_0+0) + H(x_0-0)\right].$$

$f(x) = x \quad [0, \pi]$ FCS

Easy integration $\Rightarrow$

$$x \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{m \text{ odd}} \frac{\cos mx}{m^2}$$ (p. 381)

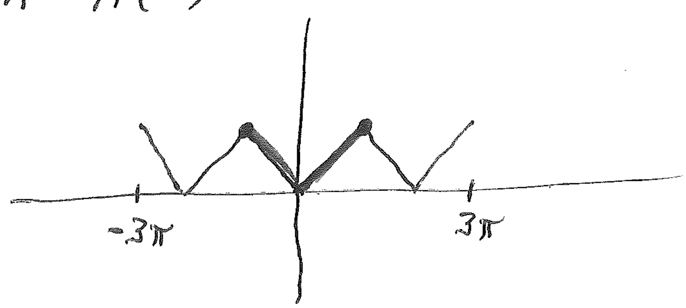
$f_{even}(x) = |x|, \; -\pi \leq x \leq \pi$



by inspection

for feven

Need 2π-periodic extension. Can go with $H(x)$:



$\left\{ \text{I use value } f(L) = f(\pi) = \pi \text{ at } x = \pi. \right\}$

This choice leads to no discontinuities. So, the graph of $\frac{\pi}{2} - \frac{4}{\pi} \sum_{m \text{ odd}} \frac{\cos mx}{m^2}$ matches $H(x)$ on $-3\pi \leq x \leq 3\pi$ (say).

Look at text, p. 44, Thm 2
very carefully.

It only references $0 < x < L$. No endpts

Also, it uses **convention** (ii).

We now want to start doing simple operations on FS.

Especially some things with integration!

For this purpose, as well as intrinsically, a few preliminaries are very helpful. Some of you may already have seen things of this sort!

Let V be a vector space over $\mathbb{R}$ (possibly infinite dimensional). We say $\langle x, y\rangle$ is an inner product on V when:

(a) $\langle x, y\rangle$ is defined for all $x \in V$, $y \in V$ {and is some real number}

(b) $\langle x, x\rangle \geq 0$ for every $x \in V$ ✓

(c) $\langle x, y\rangle = \langle y, x\rangle$ for every $x \in V$, $y \in V$

(d) $\langle cx, y\rangle = c\langle x, y\rangle$ for $c \in \mathbb{R}$

(e) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

Eg $V = \mathbb{R}^m$, $\langle \vec{x}, \vec{y}\rangle = \vec{x}\cdot\vec{y}$.

Eg $V = \{f : f$ is continuous on $[a, b]\}$

<u>or</u>

$V = \{f : f$ is <u>piecewise</u> continuous on $[a, b]\}$

and where

$$\langle f, g\rangle = \int_a^b f(x)\,g(x)\,dx \; .$$

$\sigma(x)\,dx$

Customary to define

$$\|x\| = \sqrt{\langle x, x\rangle} \; . \qquad (\geqq 0)$$

Think, e.g., $V = \mathbb{R}^m$.