From Wed.

We now want to start doing simple operations on FS.

Especially some things with integration!

For this purpose, as well as intrinsically, a few preliminaries are very helpful. Some of you may already have seen things of this sort!

Let V be a vector space over $\mathbb{R}$ (possibly infinite dimensional). We say $\langle x, y \rangle$ is an **inner product** on V when:

(a) $\langle x, y \rangle$ is defined for all $x \in V$, $y \in V$

{and is some real number}

(b) $\langle x, x \rangle \geqq 0$ for every $x \in V$ ✓

(c) $\langle x, y \rangle = \langle y, x \rangle$ for every $x \in V$, $y \in V$

(d) $\langle cx, y\rangle = c\langle x, y\rangle$ for $c \in \mathbb{R}$

(e) $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$.

Eg $V = \mathbb{R}^m$, $\langle \vec{x}, \vec{y}\rangle = \vec{x}\cdot\vec{y}$.

Eg $V = \{f : f$ is continuous on $[a,b]\}$

or

$V = \{f : f$ is piecewise continuous on $[a,b]\}$

and where

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx \quad .$$

$\sigma(x)\,dx$

Customary to define

$$\|x\| = \sqrt{\langle x, x\rangle} \quad . \qquad (\geqq 0)$$

Think, e.g., $V = \mathbb{R}^m$.

Theorem (Cauchy-Schwarz inequality)

$$|\langle x, y\rangle| \leqq \|x\|\,\|y\| \quad \text{for each } \begin{matrix} x \in V \\ y \in V \end{matrix}.$$

Pf

By properties (a) - (e), notice that:

$$Q(t) \equiv \langle tx+y, tx+y\rangle \geqq 0 \qquad -\infty < t < \infty$$

$$\Rightarrow t^2\langle x,x\rangle + 2t\langle x,y\rangle + \langle y,y\rangle \geqq 0, \text{ all } t$$

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and $R(u) \equiv \langle x+uy, x+uy\rangle \geqq 0 \qquad -\infty < u < \infty$

$$\Rightarrow u^2\langle y,y\rangle + 2u\langle x,y\rangle + \langle x,x\rangle \geqq 0, \text{ all } u.$$

If $\langle x,x\rangle = 0$ and $\langle y,y\rangle = 0$, notice that we get

$$2t\langle x,y\rangle \geqq 0, \quad 2u\langle x,y\rangle \geqq 0$$

for all t, u. Hence $\langle x,y\rangle = 0$.

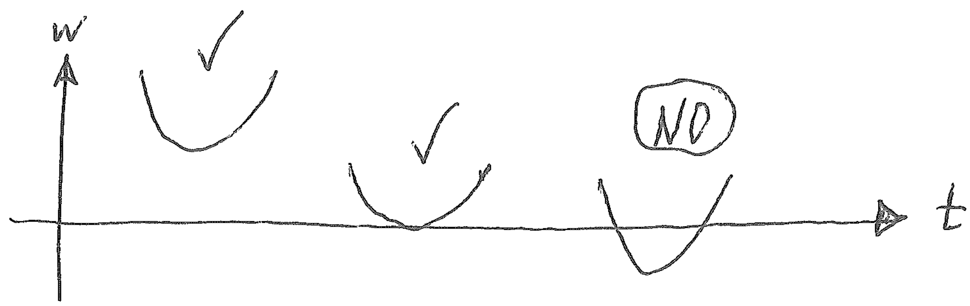
So, C-S is OK here.

Suppose next that $\langle x, x\rangle > 0$. Let $A = \langle x, x\rangle$, $B = \langle x, y\rangle$, $C = \langle y, y\rangle$. Have:

$$A\underline{t}^2 + 2B\underline{t} + C \geq 0, \text{ all } \underline{t}.$$

positive

Visualize $w = At^2 + 2Bt + C$ as a **parabola** in tw-plane.



Note $w = 0 \Leftrightarrow t = \dfrac{-B \pm \sqrt{B^2 - AC}}{A}$.

We thus CAN**NOT** have $B^2 - AC > 0$. So, $B^2 \leq AC$ and $|B| \leq \sqrt{A}\sqrt{C}$. C-S is OK.

The case where $\langle y, y \rangle > 0$ is handled with $R(u) \geqq 0$, all u. (3)

QED

So, for instance,

$$\left| \sum_{j=1}^{m} a_j b_j \right| \leqq \sqrt{\sum_{j=1}^{m} a_j^2} \sqrt{\sum_{j=1}^{m} b_j^2} .$$

Classical C–S inequality.

<u>Theorem</u> (Minkowski's Inequality)

$$\|x+y\| \leqq \|x\| + \|y\|, \quad \text{each } \begin{matrix} x \in V \\ y \in V \end{matrix}.$$

<u>Pf</u>

Must verify

$$\|x+y\|^2 \leqq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2.$$

<u>But</u>

$$\|g\|^2 = \langle g, g \rangle \quad \text{every } g \in V.$$

Must verify

$$\langle x+y, x+y \rangle \leqq \langle x, x \rangle + 2\|x\|\|y\| + \langle y, y \rangle.$$

Apply (a) – (e) again. NEED TO GET:

$$\underline{\langle x, x \rangle} + 2\langle x, y \rangle + \underline{\langle y, y \rangle}$$
$$\leqq \underline{\langle x, x \rangle} + 2\|x\|\|y\| + \underline{\langle y, y \rangle}.$$

I.e., need to check

$$\langle x, y\rangle \leq \|x\|\,\|y\| .$$

But this is OK by C-S. QED!

Classical Minkowski on $\mathbb{R}^m$:

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

{triangle inequality}

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f^2\,dx}\,\sqrt{\int_a^b g^2\,dx}$$

$$\sqrt{\int_a^b [f(x)+g(x)]^2\,dx} \leq \sqrt{\int_a^b f^2\,dx} + \sqrt{\int_a^b g^2\,dx}$$

on $C[a,b]$ or $PC[a,b]$

SERIES.

Absolute Convergence of $\sum_{n=1}^{\infty} c_n$ means

$$\sum_{n=1}^{\infty} |c_n| < +\infty .$$

Dominated Convergence of $\sum_{n=1}^{\infty} c_n(t)$ **for $t \in$ set E** means that we can find numbers M_n so that

$$|c_n(t)| \leq M_n , \quad \text{all } t \in E \ ;$$

$$\sum_{n=1}^{\infty} M_n < +\infty .$$

Note: When $|c_n(t)| \leq M_n$ for $t \in E$, we call M_n a **majorant** of $c_n(t)$ on set E.

$E = \mathbb{R}$ $\quad c_n(t) = a_n \cos nt + b_n \sin nt$

Think $M_n = |a_n| + |b_n|$.

Or $M_n = \sqrt{a_n^2 + b_n^2}$.

Uniform Convergence of $S(t) \equiv \sum_{n=1}^{\infty} c_n(t)$ **for $t \in$ set E** means that, for every $\varepsilon > 0$, we can find an integer N_ε so that

$$|S_N(t) - S(t)| < \varepsilon$$

holds at every $t \in E$ whenever $N \geqq N_\varepsilon$.

(N_ε must **not** depend on t)

Uniform convergence can be tricky to prove. There are certain tests. More on this later.

For now, we just mention

Weierstrass M_n-test.

Theorem

Given $S(t) \equiv \sum_{n=1}^{\infty} c_n(t)$ for $t \in$ set E.

Let M_n be a majorant for $c_n(t)$ on E.

Suppose that one has $\sum_{n=1}^{\infty} M_n < +\infty$.

The series $\sum_{n=1}^{\infty} c_n(t)$ then manifests

both dominated and uniform

convergence on E.

$\boxed{\text{Dominated} \Rightarrow \text{Uniform.}}$

Pf

Take $L > N$. Notice that

$$S_L(t) - S_N(t) = \sum_{N+1}^{L} c_n(t), \quad t \in E$$

$$|S_L(t) - S_N(t)| \leqq \sum_{N+1}^{L} |c_n(t)|, \quad t \in E$$

$\{\text{let } L \to \infty\}$

$$|S(t) - S_N(t)| \leqq \sum_{n=N+1}^{\infty} |c_n(t)|, \quad t \in E$$

$$\Rightarrow |S(t) - S_N(t)| \leqq \sum_{n=N+1}^{\infty} M_n, \quad t \in E.$$

For each $\varepsilon > 0$, select N_ε so big that

$$\sum_{n=N_\varepsilon+1}^{\infty} M_n < \varepsilon.$$

Clearly, we get:

$$|S(t) - S_N(t)| < \varepsilon$$

anytime $t \in E$ and $N \geqq N_\varepsilon$. QED

Example

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos nt + \frac{1}{n^3} \sin nt \right)$$

is uniformly convergent on $\mathbb{R}$. In fact, it manifests dominated convergence with $M_n = \frac{1}{n^2} + \frac{1}{n^3}$.

Similarly for

$$\sum_{n=3}^{\infty} \frac{\cos(t\sqrt{n})}{n(\ln n)^2} .$$

$$M_n = \frac{1}{n(\ln n)^2} , \quad \sum_{n=3}^{\infty} M_n < \infty ,$$

$$\int_3^{\infty} \frac{1}{x(\ln x)^2}\, dx = \int_{\ln 3}^{\infty} \frac{dv}{v^2} = \frac{1}{\ln 3} < \infty$$

{integral test}

Next — MORE ON ORTHOGONAL.

Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be any list of orthogonal functions on $[a, b]$.

Think

$$\left\{ \begin{array}{l} \langle f, g \rangle = \int_a^b f(x) g(x)\, dx \\ \text{piecewise continuous } f, g \end{array} \right\}.$$

Also $\overset{\text{concepts}}{\overline{\text{examples}}}$ 1, 2, 3, $3\frac{1}{2}$.

Let $f(x) \sim \sum_{k=1}^{\infty} c_k \varphi_k(x)$ with $c_k = \dfrac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle}$

be the "general Fourier series" associated with f. Let

$$S_N(x) = \sum_{n=1}^{N} c_n \varphi_n(x).$$

Very Basic Facts

(i) $f - S_N$ is orthogonal to $\{\varphi_1, \varphi_2, \cdots, \varphi_N\}$;

(ii) $f - S_N$ is orthogonal to **every linear combination** $\sum_{k=1}^{N} t_k \varphi_k$.

Proof

Take any ℓ, $1 \leq \ell \leq N$. Notice that

$$\langle f - S_N, \varphi_\ell \rangle = \langle f, \varphi_\ell \rangle - \langle S_N, \varphi_\ell \rangle$$

$$= \langle f, \varphi_\ell \rangle - \left\langle \sum_{n=1}^{N} c_n \varphi_n, \varphi_\ell \right\rangle$$

$$= \langle f, \varphi_\ell \rangle - \sum_{n=1}^{N} c_n \langle \varphi_n, \varphi_\ell \rangle$$

$\{$ by (a)–(e) for $\langle , \rangle$ $\}$

$$= \langle f, \varphi_\ell \rangle - 0 - c_\ell \langle \varphi_\ell, \varphi_\ell \rangle - 0$$

$$= 0. \quad \text{Aha!}$$

This proves (i). √√

For (ii), simply observe that

$$\langle f - s_N, \sum_{k=1}^{N} t_k \varphi_k \rangle$$

$$= \sum_{k=1}^{N} t_k \langle f - s_N, \varphi_k \rangle \qquad 1 \leq k \leq N$$

{by (a) – (e) for ⟨ , ⟩}

$$= \sum_{k=1}^{N} t_k \cdot 0 = 0,$$

thanks to (our) assertion (i) just proved!

QED

2^{nd} Very Basic Fact.

In the above set-up,

$$\|f\|^2 = \|f - S_N\|^2 + \sum_{j=1}^{N} c_j^2 \langle \varphi_j, \varphi_j \rangle .$$

Pf

Fix N. Let $Q = f - S_N$. By assertion (ii) above, $\langle Q, S_N \rangle = 0$. We therefore get

$$f = Q + S_N \qquad \text{"} Q \perp S_N \text{"}$$

$$\langle f, f \rangle = \langle Q + S_N, Q + S_N \rangle$$

$$= \langle Q, Q \rangle + 2\langle Q, S_N \rangle + \langle S_N, S_N \rangle$$

$$= \|Q\|^2 + 0 + \langle S_N, S_N \rangle \longleftarrow \|S_N\|^2$$

$$= \|Q\|^2 + \left\langle \sum_1^N c_n \varphi_n, \sum_1^N c_k \varphi_k \right\rangle$$

$$= \|Q\|^2 + \sum_{n=1}^{N} \sum_{k=1}^{N} c_n c_k \langle \varphi_n, \varphi_k \rangle$$

$\nwarrow$ 0 for $n \neq k$

$$= \|Q\|^2 + \sum_{n=1}^{N} c_n^2 \langle \varphi_n, \varphi_n \rangle$$

$$\Downarrow$$

$$\|f\|^2 = \|f - S_N\|^2 + \sum_{n=1}^{N} c_n^2 \langle \varphi_n, \varphi_n \rangle$$

QED

Let $N \to \infty$.

Corollary (Bessel's Inequality)

$$\sum_{j=1}^{\infty} c_j^2 \langle \varphi_j, \varphi_j \rangle \leq \|f\|^2 < +\infty.$$

We will find that, for our "classical" cases 1, 2, 3, $3\frac{1}{2}$, we actually have

$$\sum_{j=1}^{\infty} c_j^2 \langle \varphi_j, \varphi_j \rangle = \|f\|^2.$$

Much harder.

NEED A PRELIMINARY...!

Very good class of functions!

We say $f(x)$ is of **type (abc)** on $[a, a+2\pi]$ when

(a) $f(x)$ is continuous on $[a, a+2\pi]$

(b) $f(x)$ is piecewise C^1 on $[a, a+2\pi]$

(c) $f(a) = f(a+2\pi)$ $\{f$ wants to be 2π-periodic$\}$.

Similarly for $[a, a+2L]$.

Example

$[\alpha, \alpha + 2L]$ $\quad f(x)$

Type (abc)

(a) f is continuous ;

(b) f is piecewise C' ;

(c) $f(\alpha) = f(\alpha + 2L)$.

Note that the $2L$-periodic extension F can be chosen to be continuous for all $-\infty < x < \infty$.

N.B. $\{\alpha + 2kL\}$

Theorem

Let $f(x)$ be type (abc). Form FS(f):

$$f \sim \frac{1}{2} A_0 + \sum_{1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right).$$

Then:
$$\sum_{n=1}^{\infty} \left(|A_n| + |B_n| \right) < +\infty .$$

[continued]

So, we have

$$F(x) = \frac{1}{2}A_0 + \sum_{1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

with dominated + uniform convergence on all of $\mathbb{R}$.

This theorem is very basic and very important. We want to prove it.

See book pp. 47 ~ 48 "A Lemma"

That $F(x) = FS(f)$ (numerically) is obvious by Fourier's theorem and the continuity (no jumps) of F.

We need to show $\sum_{n=1}^{\infty}(|A_n| + |B_n|) < +\infty$.

USES A TRICK!

Once we do that, we can take

$$M_n = |A_n| + |B_n|$$

in the Weierstrass M_n-test; this will give dominated + uniform conv on $E = \mathbb{R}$.

We study A_n and B_n with a trick!
Assume $L = \pi$ for simplicity.

piecewise C^1

$$f \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

$$f' \overset{?}{\sim} 0 + \sum_{n=1}^{\infty} (nB_n \cos nx - nA_n \sin nx)$$

One wonders: can this be right?

It is — if f is type (abc)!