$[\alpha, \alpha + 2L]$ $f(x)$

Type (abc)

(a) f is continuous ;

(b) f is piecewise C^1 ;

(c) $f(\alpha) = f(\alpha + 2L)$.

Note that the $2L$-periodic extension F can be chosen to be continuous for all $-\infty < x < \infty$.

N.B. $\{\alpha + 2kL\}$

Theorem

Let $f(x)$ be type (abc). Form $FS(f)$:

$$f \sim \frac{1}{2} A_0 + \sum_{1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right).$$

Then:
$$\sum_{n=1}^{\infty} \left(|A_n| + |B_n| \right) < +\infty .$$

[continued]

So, we have

$$F(x) = \frac{1}{2}A_0 + \sum_{1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

with <u>dominated</u> + <u>uniform</u> <u>convergence</u> on <u>all</u> of $\mathbb{R}$.

This theorem is very basic and very important. We want to prove it.

See book pp. 47 ~ 48
"A Lemma"

That $F(x) = FS(f)$ (numerically) is obvious by <u>Fourier's</u> <u>theorem</u> and <u>the</u> <u>continuity</u> (no jumps) <u>of</u> <u>F</u>.

We need to show $\displaystyle\sum_{n=1}^{\infty}(|A_n| + |B_n|) < +\infty$.

USES A TRICK!

Once we do that, we can take

$$M_n = |A_n| + |B_n|$$

in the Weierstrass M_n-test; this will give dominated + uniform conv on $E = \mathbb{R}$.

We study A_n and B_n with a trick! Assume $L = \pi$ for simplicity.

piecewise C^1

$$f \sim \frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

$$f' \overset{?}{\sim} 0 + \sum_{n=1}^{\infty} (n B_n \cos nx - n A_n \sin nx)$$

One wonders: can this be right?

It is — if f is type (abc)!

New Stuff.

$f'(x)$ is piecewise continuous;

$f(x)$ is continuous.

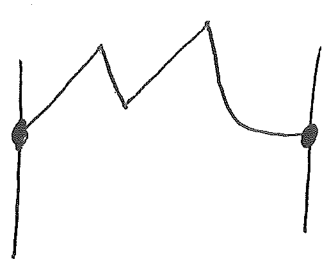
f type (abc)
$[\alpha, \alpha + 2\pi]$

Must check:

$$a_0(f') = 0$$

$$a_n(f') = n B_n, \quad n \geq 1$$

$$b_n(f') = -n A_n, \quad n \geq 1.$$



Think: **integration by parts !!!!**

$$\int_A^B u(t)v'(t)dt = u(t)v(t)\Big]_A^B - \int_A^B v(t)u'(t)dt$$

u, v
continuous +
piecewise C^1

$n \gtrsim 0$

$$a_n(f') = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f' \cos(nt)\,dt$$

$$= \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} \cos(nt)\, d[f(t)]$$ parts set-up

$$= \frac{1}{\pi}\Big[\big[\cos(nt) f(t)\big]_\alpha^{\alpha+2\pi}$$

$$- \int_\alpha^{\alpha+2\pi} f(t)\, d[\cos(nt)] \Big]$$

$$\left\{ \begin{array}{l} \cos(nt) \ \ 2\pi\text{-periodic} \\ f(\alpha) = f(\alpha+2\pi) \quad \text{KEY!!} \end{array} \right\}$$

$$= -\frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(t)\, d[\cos(nt)]$$

$$= \frac{n}{\pi}\int_\alpha^{\alpha+2\pi} f(t)\sin(nt)\,dt$$

$$= n B_n .$$ Yes!!

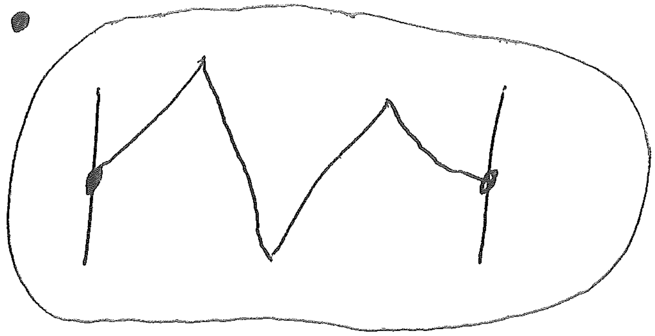
$n=0$ OK.

Note:

$$a_0(f') = \frac{1}{\pi}\int_a^{a+2\pi} f'(t)\,dt$$

$$= \frac{1}{\pi}\left[f(a+2\pi) - f(a)\right] = 0$$

by baby calc.



$\boxed{\text{use 4 chunks}}$

$n \geq 1$

$$b_n(f') = \frac{1}{\pi}\int_a^{a+2\pi} f'(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\int_a^{a+2\pi} \sin(nt)\,d[f(t)] \quad \begin{pmatrix}\text{parts}\\ \text{set-up}\end{pmatrix}$$

$$= \frac{1}{\pi}\left[\, \Big[f(t)\sin(nt)\Big]_a^{a+2\pi} - \int_a^{a+2\pi} f(t)\,d[\sin(nt)] \right]$$

$$\left\{ \begin{array}{l} \sin(nt) \quad 2\pi\text{-periodic} \\ f(a) = f(a+2\pi) \end{array} \right\}$$

$$= 0 - \frac{n}{\pi} \int_a^{a+2\pi} f(t) \cos(nt)\, dt$$

$$= -n A_n .$$

Yes!

Clearly, for this, we needed $a + b + c$.

(Otherwise it's <u>not</u> right!)

So,

$$a_n(f') = n B_n \qquad n \geq 0$$

$$b_n(f') = -n A_n \qquad n \geq 1 .$$

$$\boxed{\begin{aligned} B_n &= \frac{a_n(f')}{n} & n \geq 1 \\ A_n &= -\frac{b_n(f')}{n} & n \geq 1 \end{aligned}}$$

Can now use Cauchy-Schwarz inequality and Bessel's inequality!

Bessel's ineq.

$$g \sim \sum_1^\infty c_n \varphi_n$$

$$\|g\|^2 \geq \sum_1^\infty c_n^2 \langle \varphi_n, \varphi_n \rangle$$

Context $3\frac{1}{2}$

g on $[\alpha, \alpha + 2\pi]$, $FS(g)$

$$\int_\alpha^{\alpha+2\pi} \begin{matrix} \cos^2(kt) \\ \sin^2(kt) \end{matrix} \, dt = \pi, \quad k \geq 1$$

$$\int_\alpha^{\alpha+2\pi} 1^2 \, dt = 2\pi \qquad k = 0$$

So, Bessel's ineq $\Rightarrow$

$$\int_\alpha^{\alpha+2\pi} g(t)^2 dt \geq \left[\frac{a_0(g)}{2}\right]^2 2\pi + \sum_{k=1}^{\infty} \left(a_k(g)^2 + b_k(g)^2\right) \pi$$

$2L$ Get:

$$\int_\alpha^{\alpha+2L} g(t)^2 dt \geq \left[\frac{a_0(g)}{2}\right]^2 2L + \sum_{k=1}^{\infty} \left(a_k(g)^2 + b_k(g)^2\right) L$$

Suggestion:

Be sure you can write out Bessel's inequality QUICKLY

for

FSS on $[0, L]$

FCS on $[0, L]$

FS on $[a, a+2L]$.

$$\left\{ \begin{aligned} & f \sim \sum_{1}^{\infty} c_n \varphi_n , \\ & \|f\|^2 \geq \sum_{n=1}^{\infty} c_n^2 \langle \varphi_n, \varphi_n \rangle \end{aligned} \right\}$$

book = "THIN ICE"

Quick Way! "on the fly"

$\{\varphi_n\}_{n=1}^{\infty}$ **orthog** on $[a,b]$

$f \sim \sum_{n=1}^{\infty} c_n \varphi_n$, $S_N = \sum_{n=1}^{N} c_n \varphi_n$.

$$\|f\|^2 = \|f - S_N\|^2 + \|S_N\|^2 \quad \text{"} f - S_N \perp S_N \text{"}$$

but $\|S_N\|^2 = \sum_{n=1}^{N} \|c_n \varphi_n\|^2$ $c_n\varphi_n$'s are orthog!

So,

$$\|f\|^2 \geqq \sum_{n=1}^{N} \|c_n \varphi_n\|^2 . \quad \underline{\text{each } N}$$

$N \to \infty$

So,

$$\boxed{\|f\|^2 \geqq \sum_{n=1}^{\infty} \|c_n \varphi_n\|^2 .}$$

BESSEL'S INEQ

because

$$\langle c_n\varphi_n, c_n\varphi_n \rangle \equiv c_n^2 \langle \varphi_n, \varphi_n \rangle .$$

SLICK!

As we had before —

$g = f'$ on $[\alpha, \alpha + 2\pi]$

Get: by Bessel's ineq p. 6

$$\pi \sum_{k=1}^{\infty} \left(a_k(f')^2 + b_k(f')^2 \right) < +\infty$$

So, p. 5

$$\sum_{n=1}^{\infty} |B_n| = \sum_{n=1}^{\infty} \left| \frac{a_n(f')}{n} \right|$$

C-S

$$\leq \sqrt{\sum_{n=1}^{\infty} a_n(f')^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$$< +\infty$$

OK

And,

$$\sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} \left| \frac{-b_n(f')}{n} \right|$$

C-S

$$\leq \sqrt{\sum_{n=1}^{\infty} b_n(f')^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

$$< +\infty$$

OK

So, we see that

$$\sum_{n=1}^{\infty} \left(|A_n| + |B_n| \right) < +\infty .$$

QED

Very nice proof. $\blacksquare$

Change the subject slightly.

Some properties of (general) uniformly convergent infinite series $\sum_{n=1}^{\infty} c_n(t)$.

$\boxed{\text{Book p. 50 about 70\% down}}$

$$s(t) = \sum_{n=1}^{\infty} c_n(t) \quad \text{UNIF CONV.}$$

(A) Continuity of s

(B) Integration of s

Theorem (Continuity)

Let $E = \{a \leq t \leq b\}$, for instance.

Let $c_n(t)$ be **continuous** on E.

Let
$$S(t) = \sum_{n=1}^{\infty} c_n(t)$$
be **uniformly** convergent on E.

Then:
$$S(t) \text{ is (also) continuous on } E.$$

Proof

Choose any t_0 in E. Choose any $\varepsilon > 0$. Find N_ε so big that
$$|S_N(t) - S(t)| < \frac{\varepsilon}{3}$$
for all t in E and $N \geq N_\varepsilon$.

Freeze N to be, say, N_ε.

Know:

$$|u+v+w| \leqq |u| + |v| + |w| \;;$$

$$|S(t) - S(t_0)| = |S(t) - S_N(t) + S_N(t) - S_N(t_0) + S_N(t_0) - S(t_0)|$$

$$|S(t) - S(t_0)| \leqq |S(t) - S_N(t)| + |S_N(t) - S_N(t_0)| + |S_N(t_0) - S(t_0)|$$

$$< \frac{\varepsilon}{3} + |S_N(t) - S_N(t_0)| + \frac{\varepsilon}{3}.$$

But, $S_N(t)$ is CONTINUOUS at t_0.

Hence:

$$|S_N(t) - S_N(t_0)| < \frac{\varepsilon}{3} \quad \text{for} \quad |t - t_0| < \delta.$$

$\boxed{\text{THIS } \delta \text{ MAY DEPEND ON } t_0.}$

So, **whenever** $|t - t_0| < \delta$ $(t \in E)$,
we get

$$|S(t) - S(t_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This means $S(t)$ is continuous at t_0.
In other words, $S(t)$ is continuous on E. **QED**

Note:

You can use any kind of set for E. The interval $[a, b]$ is just the most common.

Theorem (Classical Integration Thm)

Let $E = \{a \leqq t \leqq b\}$. Let $c_n(t)$ be continuous on E and the infinite series

$$S(t) \equiv \sum_{n=1}^{\infty} c_n(t)$$

be uniformly convergent on E. Let $p(t)$ be ANY piecewise continuous function on E. Then:

$$(\star) \quad \int_a^x p(t) S(t)\, dt = \sum_{n=1}^{\infty} \int_a^x p(t) c_n(t)\, dt$$

holds for every $x \in E$; in fact, the RHS converges uniformly on E.

Caution

($\star$) can fail even for $p(t) \equiv 1$ and C^∞ functions $c_n(t)$ if you just have ordinary convergence, but NOT uniform convergence, in the original series

$$\sum_{n=1}^{\infty} c_n(t) .$$

Proof

Let $\ell = b-a$. Choose $M > 0$ so that $|p(t)| \leqq M$. Choose any $\varepsilon > 0$. By **uniform** convergence, find N_ε so big that

$$|S(t) - S_N(t)| < \frac{\varepsilon}{2\ell M}$$

for all $t \in E$ and $N \geqq N_\varepsilon$.

Notice that   X in E

$$\left| \int_a^x p(t) S(t)\,dt - \sum_{k=1}^{N} \int_a^x p(t) c_k(t)\,dt \right|$$

$$= \left| \int_a^x p(t) S(t)\,dt - \int_a^x p(t) S_N(t)\,dt \right|$$

$$= \left| \int_a^x p(t) [S(t) - S_N(t)]\,dt \right|$$

$$\leqq M \int_a^x |S(t) - S_N(t)|\,dt$$

$$\leq M \int_a^b |s(t) - s_N(t)|\,dt.$$

Keep $N \geq N_\varepsilon$. In **that** case, we get

$$\leq M \int_a^b \frac{\varepsilon}{2\ell M}\,dt$$

$$= M \frac{\varepsilon}{2\ell M}(\ell) = \frac{\varepsilon}{2} < \varepsilon.$$

This proves that

$$\sum_{k=1}^{\infty} \int_a^x p(t)\,c_k(t)\,dt$$

converges **uniformly** to $\int_a^x p(t)\,s(t)\,dt$ for $x \in E$.

QED

Real Quick Now

$\{\varphi_k\}_{k=1}^{\infty}$ orthogonal on $[a,b]$.

$\varphi_k(t)$ continuous.

EG type (abc) for FS

Suppose $f(t) \equiv \sum_{k=1}^{\infty} c_k \varphi_k(t)$ is

UNIFORMLY convergent on $[a,b]$.

$p(t)$ $f(t)$ ← ok

use Integration Thus!

$$\int_a^b f(t)f(t)\,dt = \int_a^b f(t)\Big[\sum_{k=1}^{\infty} c_k\varphi_k(t)\Big]dt$$

$\langle f, f\rangle$

$$= \sum_{k=1}^{\infty}\int_a^b f(t)\,c_k\varphi_k(t)\,dt$$

$$= \sum_{k=1}^{\infty} c_k \langle f, \varphi_k\rangle$$

$$\boxed{c_k = \frac{\langle f, \varphi_k\rangle}{\langle \varphi_k, \varphi_k\rangle}}$$

$$= \sum_{k=1}^{\infty} c_k \cdot c_k\langle \varphi_k, \varphi_k\rangle$$

$$= \sum_{k=1}^{\infty} c_k^2\langle \varphi_k, \varphi_k\rangle .$$

WOW!!

IE, $\boxed{\text{Bessel's with equal sign.}}$

So, we get what looks like an important thm.

<u>Thm</u> ($2L = 2\pi$) $\left\{\begin{array}{l}\text{Preliminary}\\ \text{Version}\end{array}\right\}$

Let f be <u>type (abc)</u> on $[\alpha, \alpha + 2\pi]$.

Let

$$f \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos nx + B_n \sin nx\right).$$

Then: $FS(f)$ is <u>unif</u> conv on $\mathbb{R}$, hence on $[\alpha, \alpha + 2\pi]$, AND WE GET

$$\int_{\alpha}^{\alpha + 2\pi} f^2\,dx = \left(\frac{A_0}{2}\right)^2 \cdot 2\pi + \sum_{n=1}^{\infty} A_n^2 \cdot \pi + \sum_{n=1}^{\infty} B_n^2 \cdot \pi .$$

Plug in on p. 19

Parseval's Equation

Experiment.

FS of x^2 on $[-\pi, \pi]$.

[Use FCS(g) $\equiv$ FS(g_{even})
and 381.]

$$\therefore\ x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)\ ,\ [-\pi, \pi]$$

Note: $f = x^2$ on $[-\pi, \pi]$ is type (abc).

Can take $M_n = \frac{4}{n^2}$ in Weierstrass M-test.

Clearly FS(x^2) is <u>uniformly</u> conv on $\mathbb{R}$.

Hence on $[-\pi, \pi]$. $\boxed{OK}$ agrees with 20

Apply p. 19 (bottom) <u>OR</u> p. 20 (bottom):

$$\int_{-\pi}^{\pi} (x^2)^2 dx = \left(\frac{\pi^2}{3}\right)^2 \underline{2\pi} + \sum_{n=1}^{\infty} \frac{16}{n^4}\underline{\pi} \implies$$

$$2\int_0^{\pi} x^4 dx = \frac{2\pi^5}{9} + 16\pi\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right) \implies$$

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 16\pi S,$$

$$\text{where} \quad S \equiv \sum_{n=1}^{\infty} n^{-4} \implies$$

$$2\pi^5\left(\frac{1}{5} - \frac{1}{9}\right) = 16\pi S$$

$$2\pi^5 \frac{4}{45} = 16\pi S$$

$$\frac{\pi^5}{45} = 2\pi S$$

$$\Rightarrow \quad S = \frac{\pi^4}{90}.$$

So,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

One of the most famous series in math! Euler ~ 1750