"Inspiration"

A ①

$\mathbb{R}^3$ $\mathcal{V}$ 3-dim vector space!

$$\vec{v} \cdot \vec{w} = \sum_{j=1}^{3} a_j b_j \qquad \vec{v} = \langle a_1, a_2, a_3 \rangle$$

etc

$$|\vec{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \text{length of } \vec{v}$$

So, $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2$$

$\vec{v} \perp \vec{w}$, $\vec{v}$ orthogonal to $\vec{w}$

$$\Leftrightarrow \vec{v} \cdot \vec{w} = 0$$

Recall: there are many bases for $\mathcal{V}$.

Orthogonal bases are especially nice.

TAKE:
basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ orthogonal

Write:

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 .$$

(Exactly one way.) What is c_j?

$$\vec{v} \cdot \vec{v}_1 = c_1 (\vec{v}_1 \cdot \vec{v}_1) + c_2 (\vec{v}_2 \cdot \vec{v}_1) + c_3 (\vec{v}_3 \cdot \vec{v}_1)$$

$$= c_1 (\vec{v}_1 \cdot \vec{v}_1) + 0 + 0$$

$$c_1 = \frac{\vec{v} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$$ c_j similar

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Alternate notation for dot product

$$\langle \vec{v}, \vec{w} \rangle .$$

"inner product"

So,

$$c_j = \frac{\langle \vec{v}, \vec{v_j} \rangle}{\langle \vec{v_j}, \vec{v_j} \rangle}.$$

Can do $\mathbb{R}^n$ similarly. V

$$\vec{v} = c_1 \vec{v_1} + \cdots + c_n \vec{v_n}$$

$$\Rightarrow \quad c_j = \frac{\langle \vec{v}, \vec{v_j} \rangle}{\langle \vec{v_j}, \vec{v_j} \rangle} \qquad 1 \leq j \leq n$$

Notice too: (back in $\mathbb{R}^3$)

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

$$\vec{v} \cdot \vec{v} = (c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3) \cdot (\text{same})$$

$$= \sum_{j=1}^{3} \sum_{k=1}^{3} c_j c_k (\vec{v}_j \cdot \vec{v}_k)$$

$$= 0 + \sum_{j=1}^{3} c_j c_j (\vec{v}_j \cdot \vec{v}_j)$$

So:

$$\langle \vec{v}, \vec{v} \rangle = \sum_{j=1}^{3} c_j^2 \langle v_j, v_j \rangle .$$

Similarly for $\mathbb{R}^n$.

(For unit vectors, $\vec{v}_j$'s, looks like Pythagorean theorem.)

There are many <u>other</u> <u>sorts</u> of vector spaces $\mathcal{V}$.

Eg $\mathcal{V} = C^1[a, b]$ functions f

OR

$$\mathcal{V} = \{ f \in C^1[a, b] , \ f(a) = f(b) = 0 \}.$$

"Dot product" is

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx .$$

Thus: in $\mathcal{V}$

$$f \perp g \iff \langle f, g \rangle = 0 .$$

orthogonal

Remarkable Fact

Let $a = 0$, $b = \pi$, say.

Every "vector" f in $\mathcal{V}$ can be written as an infinite series

$$f = \sum_{j=1}^{\infty} c_j f_j, \quad c_j = \frac{\langle f, f_j \rangle}{\langle f_j, f_j \rangle}$$

where $\{f_j\}_{j=1}^{\infty}$ is some given set of orthogonal "vectors".

We will see: f_j

$\cos[(j-1)x]$ $\quad \{\cos 0 = 1\}$ #1

$\sin(jx)$ #2

"Fourier Analysis is about this (idea)!"

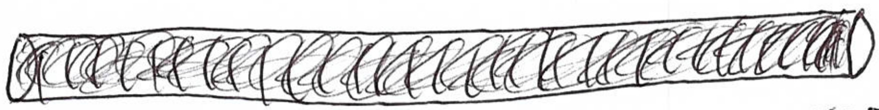
VERY USEFUL.

Math 4567

Sample Problem.

Given metal rod, $x = 0$ to $x = \pi$;
ends kept in ice water at temp 0.

Assume lateral edge insulated.



$x=0$ $x=\pi$

Let $u(x,t)$ = temperature of metal rod at position x and time $t \geqq 0$.

Assume that we know
$$u(x,0) = x(\pi - x).$$

What is temperature $u(x,t)$ for $t > 0$?

We will see:

$$(*) \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad \begin{pmatrix} \text{HEAT} \\ \text{EQ.} \end{pmatrix}$$

i.e. $u_t = k\, u_{xx}$. p.d.e.

To reflect the "end" condition (need):

$$(**) \quad u(0,t) = 0 \; ; \quad u(\pi,t) = 0 .$$

Do we $\boxed{\text{know}}$ any solutions of $(*)$ satisfying $(**)$?

YES! EASY. Infinite #

$$e^{-kn^2 t} \sin(nx) \qquad n \geq 1$$

"$A(t)\,B(x)$" format

But, then, KEY OBSERVATION,

$$u = \sum_{n=1}^{1000} b_n e^{-kn^2 t} \sin(nx)$$

also works in (*) and (**)!

LINEAR COMB.

Can we get AT $t=0$:

$$x(\pi - x) \approx \sum_{n=1}^{1000} b_n \sin(nx) \ ??$$

NO! No reason! Not likely!

(a) Why 1000?

(b) What does $\sin(nx)$ has to do with a polynomial?

Remarkable Fact.

Given **any** function $f(x)$ on $0 \leq x \leq \pi$ with $f(0) = 0$, $f(\pi) = 0$.

Assume f is "nice"; say C^1.

Then: we can write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

note ∞

for CERTAIN b_n !!

Of course, key question arises:

fine, but what does b_n equal?

Also: why convergent?

Grant me this FACT. Ok?

Look at

$$u = \sum_{n=1}^{\infty} \underline{\underline{b_n}} e^{-kn^2 t} \sin(nx) .$$

<u>Probably</u> nicely convergent for any $t > 0$.

We see that u is "rigged" to satisfy

(★) $u_t = k u_{xx}$ $0 \leq x \leq \pi$ ✓

(★★) $u(0,t) = 0$ and $u(\pi, t) = 0$ ✓

(★★★) $u(x, 0) = f(x)$ for $t = 0$. ✓

Great!

(BVP)

We will see later:

$$x(\pi - x) = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{1}{n^3} \sin(nx).$$

Page 382 [table]. Note:

$$\sum_{n \text{ odd}} \frac{1}{n^3} < \infty, \quad \text{OK!}$$

So, knowing this, we now conclude

$$u = \frac{8}{\pi} \sum_{n \text{ odd}} \frac{1}{n^3} e^{-kn^2 t} \sin(nx)$$

fits ($\star$), ($\star\star$), and

$$u(x,0) = x(\pi - x).$$

"ASSUMING EVERYTHING CONVERGENT"

Insofar as our physical problem has a unique solution, we have FOUND it!!!

Great!!

"Not very pretty". But right!

Pause! $(N \leq \infty)$ 8

$$\sum_{n=1}^{N} c_n = \sum_{m=1}^{N} c_m$$

$$= \sum_{k=1}^{N} c_k = \sum_{\ell=1}^{N} c_\ell$$

n, m, k, ℓ DUMMY VARIABLES

$n \geqq 1$ odd means $n = 2k-1$, $k \geqq 1$

$n \geqq 1$ even means $n = 2\ell$, $\ell \geqq 1$

of course.

So,

$$\sum_{n \text{ odd}} \frac{1}{n^6} \equiv \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6}$$

$$\sum_{n \text{ odd}} \frac{1}{n^6} \sin(nx) \equiv \sum_{k=1}^{\infty} \frac{1}{(2k-1)^6} \sin[(2k-1)x].$$

(etc)

In textbook, authors often use "n" on RHS. Confuses students! Bad style; but it's a DUMMY VARIABLE.

$$\boxed{\text{STAY ALERT.}}$$

Baby Trig

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Know! High School!

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

Note:

$$\sin^2(A) = \frac{1}{2}\left[1 - \cos(2A)\right]$$

$$\cos^2(A) = \frac{1}{2}\left[1 + \cos(2A)\right].$$

Know in your sleep!! $\left\{\begin{matrix}\text{Calc}\\ 1\end{matrix}\right\}$

Key Observation (> 250 yrs ago):

$$\int_0^{\pi} \sin(kx)\sin(\ell x)\,dx = \begin{cases} 0, & k \neq \ell \\ \frac{\pi}{2}, & k = \ell \end{cases}$$

$k \geq 1$
$\ell \geq 1$

AND

$$\int_{-\pi}^{\pi} \sin(kx)\sin(\ell x)\,dx = \begin{cases} 0, & k \neq \ell \\ \pi, & k = \ell \end{cases} .$$

"Very Special Property" $k \neq \ell$

Similarly $\cos(kx)\cos(\ell x)$.

For $\sin(kx)\cos(\ell x)$, stay on $[-\pi, \pi]$.

$\hookrightarrow$ get 0 !

To be 100% clear:

$$\int_0^{\pi} \cos(kx)\cos(lx)\,dx = \begin{cases} 0, & k \neq l,\ k \geq 0,\ l \geq 0 \\ \pi/2, & k = l \geq 1 \\ \pi, & k = l = 0 \end{cases}$$

multiply by 2 for $[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \sin(kx)\cos(lx)\,dx = 0 \qquad \begin{matrix} k \geq 1 \\ l \geq 0 \end{matrix}$$

NO STATEMENT FOR $[0, \pi]$

Proof of Key Observation.

$\sin(kx)\sin(lx)$ is EVEN. $\begin{pmatrix} k \geqq 1 \\ l \geqq 1 \end{pmatrix}$

Suffices to do $[0, \pi]$.

For $k = l$, use baby calc 1. OK

For $k > l$ (say),

$$\int_0^\pi \sin(kx)\sin(lx)\,dx$$

$$= \frac{1}{2}\int_0^\pi \left[\cos(k-l)x - \cos(k+l)x\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(k-l)x}{k-l} - \frac{\sin(k+l)x}{k+l}\right]_0^\pi$$

$$= \frac{1}{2}\left[0 - 0 - 0 + 0\right] = 0.$$

OK

$\cos(kx)\cos(\ell x)$?   You do it.

Use:

$$\cos(kx)\cos(\ell x) = \frac{1}{2}\left[\cos(k-\ell)x + \cos(k+\ell)x\right].$$

Finally, for $\sin(kx)\cos(\ell x)$, note that this function is **ODD**. So,

$$\int_{-\pi}^{\pi} \sin(kx)\cos(\ell x)\,dx = 0$$

obviously.

OK

Caution:

$\int_{0}^{\pi} \sin(kx)\cos(\ell x)\,dx$ is NOT zero, in general...