We had

$$\int_0^{\pi} \sin(kx)\sin(\ell x)\,dx = \begin{cases} 0, & k \neq \ell \\ \frac{\pi}{2}, & k = \ell \end{cases}$$

also

$$\int_0^{\pi} \cos(kx)\cos(\ell x)\,dx = \begin{cases} 0, & k \neq \ell \\ \pi/2, & k = \ell \geqq 1 \\ \pi, & k = \ell = 0 \end{cases}$$

The $\frac{\pi}{2}$, π stuff is just $\sin^2(A)$, $\cos^2(A)$.
No problem!!

CLAIM: The stuff with 0 has an easy proof by integ by parts.

$$\int_a^b fg'\,dx = [fg]_a^b - \int_a^b gf'\,dx$$

~~so~~

$$\int_a^b [f'g + fg']\,dx = [fg]_a^b$$

$$\left\{ \begin{array}{c} \text{Fund Thm of Integ Calc} \\ \text{applied to } fg \end{array} \right\}$$

Replace g by g'.

$$\int_a^b [\underline{f'g'} + fg'']\,dx = [fg']_a^b \qquad \blacksquare$$

Switch f and g. Get:

$$\int_a^b [\underline{f'g'} + gf'']\,dx = [gf']_a^b$$

$$\Downarrow$$

$$\int_a^b [fg'' - gf'']\,dx = [fg' - gf']_a^b \qquad \blacksquare$$

Both these highlighted identities are trivial once you write them down.

$$\int_0^\pi [fg'' - gf'']\,dx = [fg' - gf']_0^\pi$$

Take —

$f = \sin(kx)$, $g = \sin(\ell x)$

$f'' = -k^2 f$, $g'' = -\ell^2 g$ (ODE) $k \neq \ell$

Get:

$$\int_0^\pi [-\underline{\ell}^2 fg + \underline{k}^2 fg]\,dx = 0 - 0$$

$$(k^2 - \ell^2)\int_0^\pi fg\,dx = 0$$ Aha!!

$$\therefore \int_0^\pi \sin(kx)\sin(\ell x)\,dx = 0, \quad k \neq \ell$$

Same thing with

$f = \cos(kx)$, $g = \cos(\ell x)$!

$$\boxed{\begin{aligned} \text{note}\quad f'(0) &= f'(\pi) = 0 \\ g'(0) &= g'(\pi) = 0 \end{aligned}}$$

∎

⇒ QED

SLICK!! TRIVIAL!

Uses the ODE

$$f'' + k^2 f = 0$$

$$g'' + \ell^2 g = 0 \; .$$

Return now to Wednesday's lecture, with our <u>trying to</u> solve heat equation (BVP)

☆ $u_t = k u_{xx} \quad 0 \leq x \leq \pi, \; t \geq 0$

☆☆ $u(0,t) = 0 \; , \; u(\pi,t) = 0$

☆☆☆ $u(x,0) = f(x)$.

Were playing with

$$u \equiv \sum_{n=1}^{\infty} b_n e^{-kn^2 t} \sin(nx)$$

as "something interesting".

⟨ Tentative ⟩

or p. ①

Use Key Observation (there) and logic!!

Want:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

(Hope It Can Be Done)

Everything Convergent!!

EACH m.

$$\therefore \quad f(x)\sin(mx) = \sum_{n=1}^{\infty} b_n \sin(nx)\sin(mx)$$

$$\therefore \quad \int_0^{\pi} f(x)\sin(mx)\,dx = \sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin(nx)\sin(mx)\,dx$$

$$= 0 + b_m\left(\frac{\pi}{2}\right) + 0$$

So: we must have

$$\boxed{b_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx, \quad m \geq 1.}$$

Re-phrased, given any

f C^1 on $[0,\pi]$

with $f(0) = f(\pi) = 0$.

We think / suspect / hope / dream (?)
↑

that:

$$\boxed{\begin{array}{c} f(x) = \displaystyle\sum_{n=1}^{\infty} \underline{b_n} \sin(nx) \\ \text{where} \\ b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin(nx)\,dx \end{array}}$$.

WOW!

1750 Euler, ~1820 Fourier

Proof?

Let's pause.

What's so special about

$$\int_a^b g(x)h(x)dx = 0 \text{ ?}$$

We say g is orthogonal to h.

on $[a,b]$

WHY?

Analogy. $\vec{x} = \langle x_1, \dots, x_N \rangle$ $\mathbb{R}^N$

$\vec{y} = \langle y_1, \dots, y_N \rangle$

$$\vec{x} \cdot \vec{y} = \sum_{j=1}^{N} x_j y_j$$

$$\int_a^b g(t)\, h(t)\, dt$$

Riemann Sum! N giant!

$$t_j = a + j\left(\frac{b-a}{N}\right)$$

$$t_0 < t_1 < \cdots < t_N$$

Integral $\approx \displaystyle\sum_{j=1}^{N} g(t_j)h(t_j)\Delta t_j$

Integral $\approx \left[\dfrac{b-a}{N}\right] \displaystyle\sum_{j=1}^{N} g(t_j)h(t_j)$

Like a dot product in $\mathbb{R}^N$ with a fudge factor!

OK

It's kind of like the dot product is now (for functions)

$$\langle g, h \rangle = \int_a^b g(t)h(t)\,dt \,.$$

Example:

The successive functions

$$\sin x, \ \sin 2x, \ \sin 3x, \ \ldots$$

are ORTHOGONAL on $[0, \pi]$.

$$\langle \sin(kx), \sin(\ell x) \rangle = 0$$

whenever $k \neq \ell$.

Saying

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{on } [0,\pi]$$

means that f is an infinite linear combination of certain basis "vectors" which are mutually orthogonal (perpendicular).

Here: functions = "vectors"

Linear Algebra ↔ Vector Spaces

Examples? Solutions of homogeneous linear ODE

Another thing! $\boxed{u_t = k u_{xx}}$

Where did $e^{-kn^2t}\sin(nx)$ come from ?? $\boxed{u(0,t) = u(\pi,t) = 0}$

Guessing ? NO!!

Wanted $A(t)B(x)$ format

AND $B(0) = B(\pi) = 0$.

Need:

$$A'(t)\,B(x) = k\,A(t)\,B''(x)$$

or

$$\frac{1}{k}\,\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} \quad .$$

So, $$\frac{1}{k}\,\frac{A'(t)}{A(t)} = \frac{B''(x)}{B(x)} = \mathcal{C}$$

Try $\mathcal{C} < 0$. Say $\mathcal{C} = -R^2$.

so, $$B''(x) + R^2 B(x) = 0$$

$B(0) = 0$
$B(\pi) = 0$

and, $$\frac{A'(t)}{A(t)} = -kR^2$$

$$B(x) = c_1 \cos(Rx) + c_2 \sin(Rx)$$

$$B(0) = 0 \rightarrow 0 = c_1 + 0 \Rightarrow c_1 = 0$$

$$B(\pi) = 0 \rightarrow 0 = c_2 \sin(R\pi)$$

✓✓

Go with $R = \underline{n} \geq 1$.

OK! $\mathcal{C} = -n^2$, $\dfrac{A'(t)}{A(t)} = -kn^2$

So, we can go with

$$R = n \geqq 1$$

$$B(x) = C_2 \sin(nx)$$

e.g. $C_2 = 1$

and $$\frac{A'(t)}{A(t)} = -kn^2$$

e.g. $A(t) = e^{-kn^2 t}$

$$\Downarrow$$

can go with (e.g.)

$$A(t)\,B(x) = e^{-kn^2 t} \sin(nx) \; .$$

This is where it came from!

NOTE:

$$e^{-k\omega^2 t}\sin(\omega x) = A(t)B(x)$$

(still) satisfies heat equation *

for ANY $\omega > 0$

(but not necessarily 0, 0).

0 at $x=0$ is fine!

We see: $\omega = n$ is special

---

* $u_t = k u_{xx}$

Another thing! Why π?

$$0 \leq x \leq \underline{L}$$

Define:

$$y = \frac{\pi}{L} x .$$

Thus $0 \leq y \leq \pi$.

$\sin(ny)$ = interesting on $[0, \pi]$

$$\Downarrow$$

$\sin\left(n \frac{\pi}{L} x\right)$ = interesting on $[0, \underline{L}]$

Aha!!

So, we look at:

$$\begin{pmatrix} n \geqq 1 \\ m \geqq 1 \end{pmatrix}$$

$$\int_0^L \sin\left(n\frac{\pi}{L}x\right)\sin\left(m\frac{\pi}{L}x\right)dx$$

$$= \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}.$$

Pf.

Just set $y = \frac{\pi}{L}x$, $x = \frac{L}{\pi}y$ in the left-hand integral. $\boxed{\text{EASY}}$

$\Rightarrow$ get Analog of Key Observation etc.

$\sin(k\frac{\pi}{L}x)$, $\cos(\ell\frac{\pi}{L}x)$, etc etc

From Earlier Lecture

Key Observation (> 250 yrs ago):

$$\int_{\underline{0}}^{\pi} \sin(kx)\sin(lx)\,dx = \begin{cases} 0, & k \neq l \\ \frac{\pi}{2}, & k = l \end{cases}$$

$k \geqq 1$
$l \geqq 1$

AND

$$\int_{\underline{-\pi}}^{\pi} \sin(kx)\sin(lx)\,dx = \begin{cases} 0, & k \neq l \\ \pi, & k = l \end{cases} .$$

"Very Special Property" $k \neq l$

Similarly $\cos(kx)\cos(lx)$.

For $\sin(kx)\cos(lx)$, stay on $[-\pi, \pi]$.

↳ get 0 !

So, now, on —

$[0, L]$. $f(0) = 0$ $f(L) = 0$ $f \in C^1$

$$f(x) \simeq \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) \qquad \boxed{\text{hope}}$$

$\Downarrow$ EACH k

$$f(x) \sin\left(k \frac{\pi}{L} x\right) \approx \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right) \sin\left(k \frac{\pi}{L} x\right)$$

Integrate over $0 \leq x \leq L$.

$$\int_0^L f(x) \sin\left(k \frac{\pi}{L} x\right) dx = 0 + \underbrace{b_k \left(\frac{L}{2}\right)}_{\uparrow \ \text{at } n=k} + 0$$

$$\boxed{b_k = \frac{2}{L} \int_0^L f(x) \sin\left(k \frac{\pi}{L} x\right) dx}$$

Equivalently, for $[0, L]$, want:

$$b_k = \frac{\left\langle f(x), \sin\left(\frac{k\pi x}{L}\right) \right\rangle}{\underbrace{\left\langle \sin\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \right\rangle}_{\frac{L}{2}}} .$$

Note the format!

So —

Expertation (Hope)

$f(x)$ on $0 \leqq x \leqq L$

f C^1 $f(0) = f(L) = 0$

Then:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n \frac{\pi}{L} x\right)$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(n \frac{\pi}{L} x\right) dx .$$

Fourier Sine Series
Development of f(x)!
(Expansion)

{Euler, Fourier}

Next, a
Very Convenient Concept:

<u>Piecewise</u> <u>continuous</u> function $f(x)$ on $[a,b]$.

① $f(x)$ is defined at each $x \in [a,b]$.

② At <u>each</u> $w \in [a,b]$,

$$f(w+) = f(w+0) = \lim_{x \to w^+} f(x) \text{ exists;}$$

$$f(w-) = f(w-0) = \lim_{x \to w^-} f(x) \text{ exists.}$$

[Obvious modification if $w = a$ or b]

③ At all but a finite # of w, we have

$$f(w-) = f(w+) = f(w).$$

{ I.e., at all but a finite number of points w, the function f is continuous. }

Alternate — equivalent — definition.

More graphical!

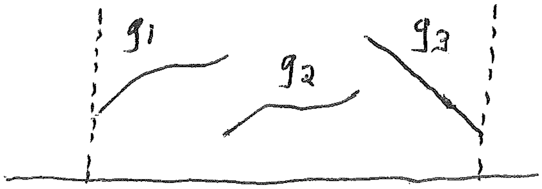
$f(x)$ is "piecewise" equivalent to $m < \infty$ continuous functions $\{g_1, g_2, \ldots, g_m\}$ "strung out" on a grid

$$[\underset{\overset{\uparrow}{a}}{x_0}, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{m-1}, \underset{\overset{\uparrow}{b}}{x_m}]$$

where $g_j \in C[x_{j-1}, x_j]$ and

$$f(x) = \left\{ \begin{array}{ll} g_1(x), & x_0 < x < x_1 \\ g_2(x), & x_1 < x < x_2 \\ \quad \cdots & \\ g_m(x), & x_{m-1} < x < x_m \end{array} \right\}.$$

Note that we don't care what f is at points $x_0, x_1, \ldots, x_m$. This is intentional.



Similarly for piecewise C^1, C^2, C^3, etc
on $[a, b]$. $g_i \in C^1$, etc

Notation: $C_p[a,b]$ or $PC[a,b]$

$C_p[a,b]$ is a vector space.

Easy! $f+g$ lump all the grid points together; i.e., refine the grid

By def, for $f \in C_p[a,b]$,

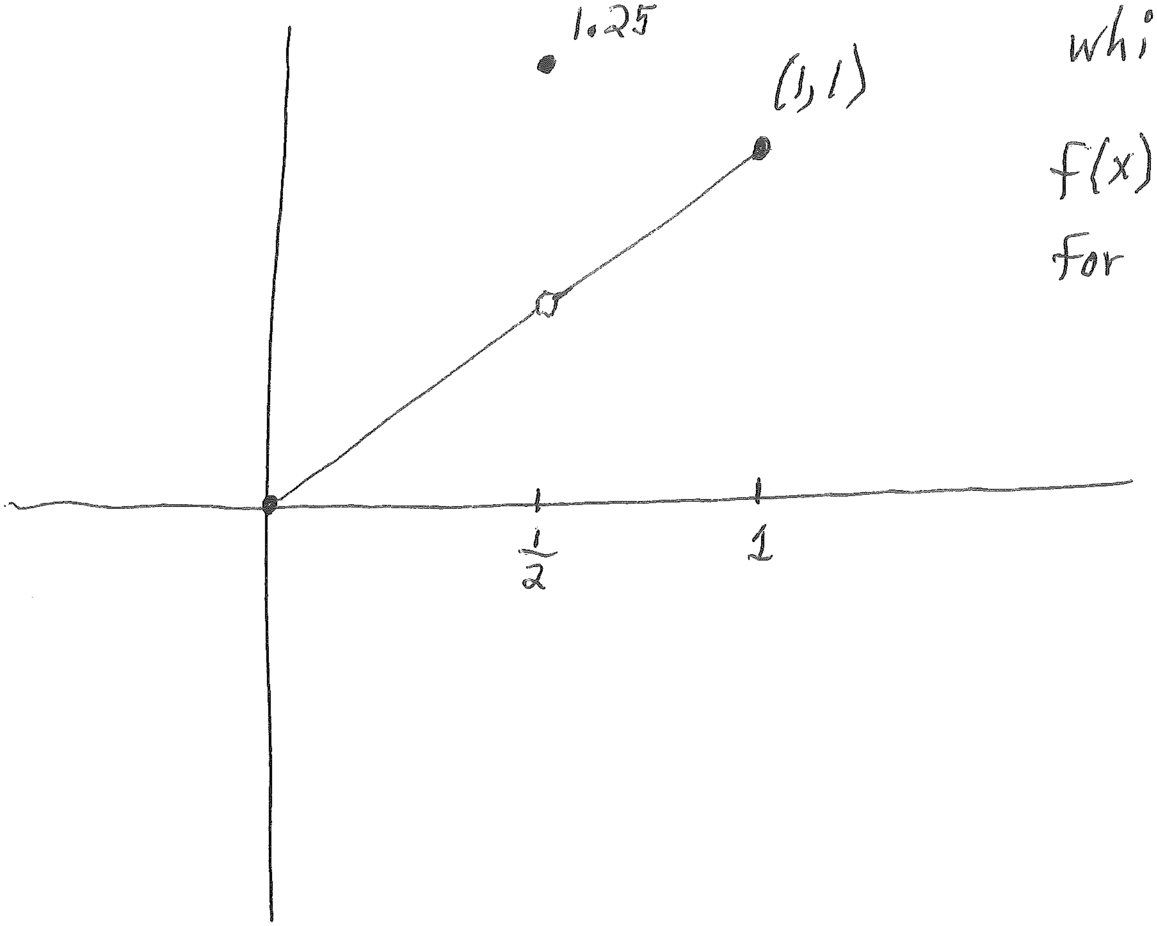
$$\int_a^b f(x)dx = \int_{x_0}^{x_1} g_1(x)dx + \cdots + \int_{x_{m-1}}^{x_m} g_m(x)dx.$$

This def is legit.

"Area"

Caution:

$$\boxed{f(\tfrac{1}{2}) = 1.25}$$



while

$f(x) = x$

for $x \neq \frac{1}{2}$

$g_1 = x$ grid $[0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$

$g_2 = x$

piecewise continuous f

$\left\{ \begin{array}{l} x = \frac{1}{2} \text{ still regarded as a} \\ \text{"jump" discontinuity} \end{array} \right\}$