"SOME BACKGROUND"

Periodic function $f(x)$ on $\mathbb{R}$:

$$f(x+\ell) = f(x), \quad \text{all } x.$$

We call f ℓ-*periodic* or periodic (ℓ).

NOTE THAT f is automatically $2\ell, 3\ell, 4\ell, \ldots$ periodic!

When f is periodic, there is typically a *least* period!!

$\sin(x)$	2π	$\tan(x)$	π
$\sin(5x)$	$\frac{2\pi}{5}$	$\tan(5x)$	$\frac{\pi}{5}$
$\sin(\omega x)$	$\frac{2\pi}{\omega}$	$\tan(\omega x+3)$	$\frac{\pi}{\omega}$
etc	etc	etc	etc

Let $f(x)$ be piecewise continuous on interval $[a,b]$. Let $b-a=\ell$ = length of interval.

We call piecewise continuous function F an ℓ-periodic extension of f when

(a) F is defined for all $x \in \mathbb{R}$;

(b) F is ℓ-periodic ($F(x+\ell)=F(x)$);

(c) F agrees with f on $a<x<b$.

Common Convention:

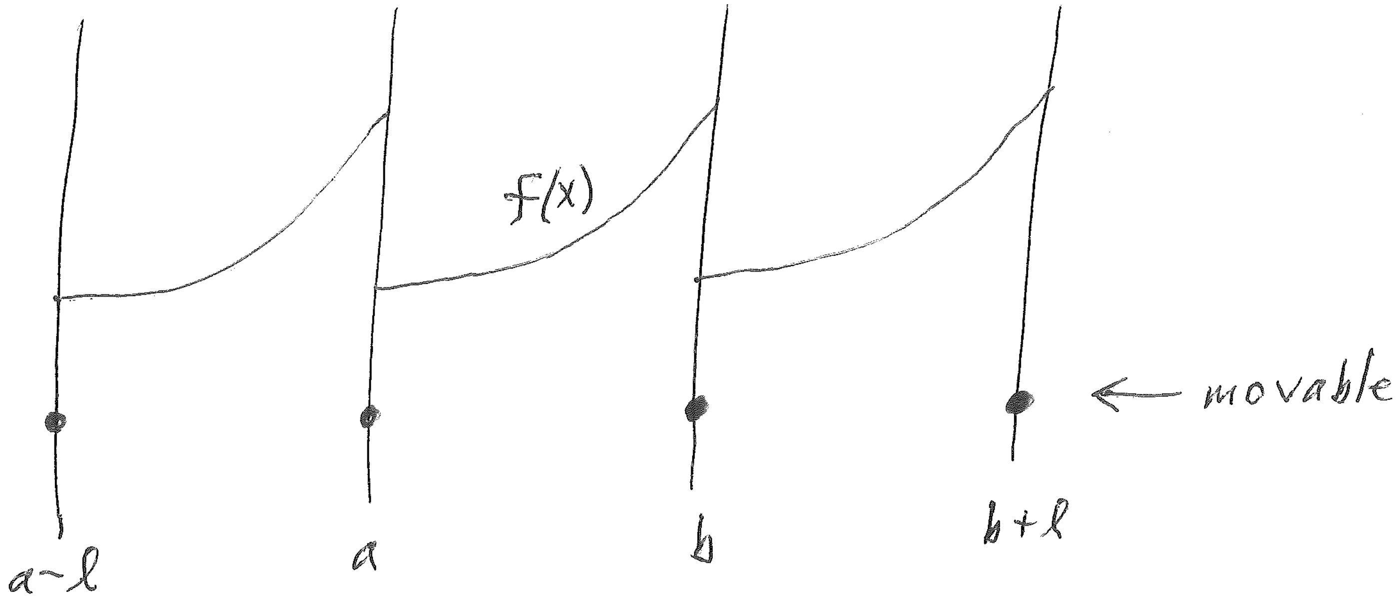
(d) if $f(a)=f(b)$ AND $x=a,b$ are both points of continuity of f, we agree that

$$F(a)=f(a)=f(b)=F(b).$$

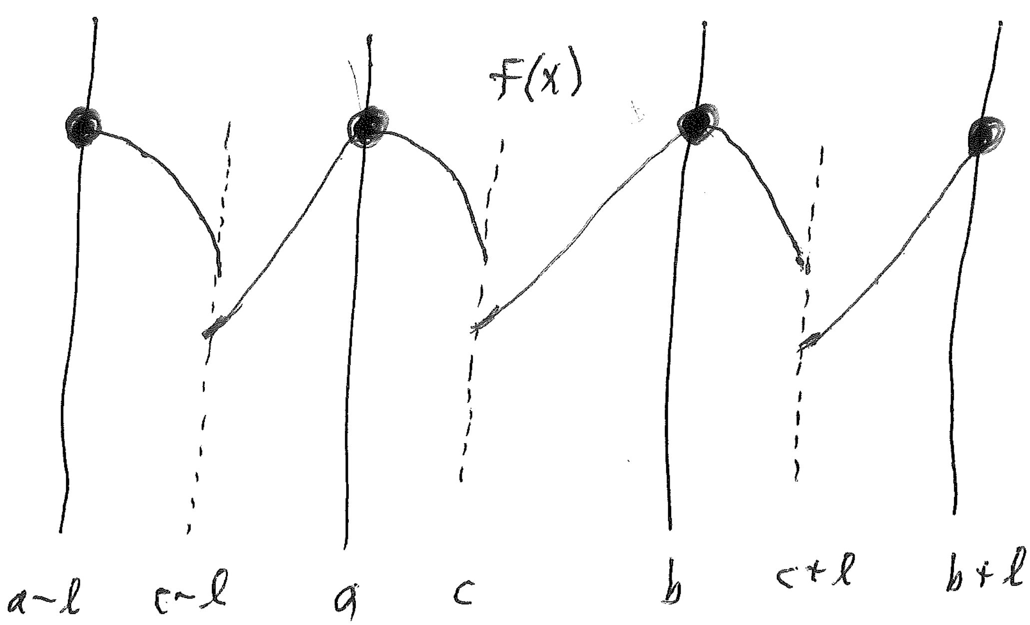
NOTE that $F(x)$ is always uniquely determined on any open interval

$$a+n\ell < x < b+n\ell, \quad n \in \mathbb{Z}.$$

$$\boxed{F(x)=f(x-n\ell)} \leftarrow \text{KEY}$$



but
(if convention used)



We continue our discussion of certain trig functions (cos and sin) and how they can be viewed as ORTHOGONAL!

We can now fill in (most of) "the pieces" of chap 1.

Recall g is <u>orthogonal</u> to h on $[a, b]$ when

$$\langle g, h \rangle \equiv \int_a^b g(x)h(x)\,dx = 0.$$

inner product

a "dot product" for functions

3 main cases right now!

②

- $[0, L]$ $\quad \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$$

CAN WRITE $\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right) \right\rangle$

- $[0, L]$ $\quad \left\{ \cos\left(\frac{k\pi x}{L}\right) \right\}_{k=0}^{\infty}$

$$\int_0^L \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{\ell\pi x}{L}\right) dx = \begin{cases} 0, & k \neq \ell \\ L/2, & k = \ell \geq 1 \\ L, & k = \ell = 0 \end{cases}$$

- $[-L, L]$ lump everything together:

$$\left\{\cos\left(\frac{k\pi x}{L}\right)\right\}_{k=0}^{\infty} \cup \left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}.$$

view as a list!

Have:

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{l\pi x}{L}\right)dx = \begin{cases} 0, & k \neq l \\ L, & k = l \geq 1 \\ 2L, & k = l = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ L, & n = m \geq 1 \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0 \quad \text{ANY } k \geq 0, n \geq 1.$$

OK, then. GIVEN **any** **list** of orthogonal functions

$$\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$$

on $a \leq x \leq b$.

Take **any*** $f(x)$ on $[a, b]$.

Hope

$$f(x) = \sum_{m=1}^{\infty} c_m \varphi_m(x),$$

at least at "most" values of x.

[For us, all but a finite # of x.]

---

* reasonable!

Get:

$$f(x)\varphi_r(x) = \sum_{m=1}^{\infty} c_m \varphi_m(x)\varphi_r(x)$$

except finite # of x

$$\Downarrow$$

$$\int_a^b f(x)\varphi_r(x)\,dx = \sum_{m=1}^{\infty} c_m \int_a^b \varphi_m(x)\varphi_r(x)\,dx$$

each r

{integrals not affected by "bad" x}

finite # of

$$\langle f,\varphi_r\rangle = \underbrace{0}_{m<r} + c_r\langle\varphi_r,\varphi_r\rangle + \underbrace{0}_{m>r}$$

$$\therefore \quad c_r = \frac{\langle f,\varphi_r\rangle}{\langle\varphi_r,\varphi_r\rangle}$$

nice!

In other words, we hope:

$$\boxed{f(x) = \sum_{r=1}^{\infty} c_r \varphi_r(x)}$$

except at finite # of x

wherein

$$\boxed{c_r = \frac{\langle f, \varphi_r \rangle}{\langle \varphi_r, \varphi_r \rangle}}$$

$$c_r = \frac{\int_a^b f(x)\varphi_r(x)\,dx}{\int_a^b \varphi_r(x)\varphi_r(x)\,dx}.$$

Note the pattern carefully!!!!

The series in $\boxed{}$ + $\boxed{}$ is called (by definition!!) the Fourier series of $f(x)$ with respect to $\{\varphi_m\}_{m=1}^{\infty}$ on $[a, b]$.

NOTE:

We do <u>not</u> know yet that, in $\boxed{} + \boxed{}$,

(A) the RHS converges;

(B) we have <u>equality</u> <u>at</u> "most" x

⟨i.e., all but finitely many⟩.

THESE THINGS <u>REQUIRE</u> PROOF!! Nontrivial.

Common to write

$$f(x) \sim \sum_{r=1}^{\infty} c_r \varphi_r(x)$$

"awaiting (A) and (B)".

Example 1 (all sines)

$[0,L]$ $\quad \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{\langle f, \varphi_n \rangle}{\langle \varphi_n, \varphi_n \rangle} = \frac{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx}$$

so

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Save It! We call this the FOURIER SINE SERIES of $f(x)$ on $[0,L]$.

Example 2 (all cosines)

$[0, L]$ $\quad \left\{ \cos\left(\frac{k\pi x}{L}\right) \right\}_{k=0}^{\infty}$ ← NOTE!

$$f(x) \sim \sum_{k=0}^{\infty} c_k \cos\left(\frac{k\pi x}{L}\right)$$

$$c_k = \frac{\langle f, \varphi_k \rangle}{\langle \varphi_k, \varphi_k \rangle} = \frac{\int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx}{\int_0^L \cos^2\left(\frac{k\pi x}{L}\right) dx}$$

$$\begin{array}{ll} k = 0 & L \\ k \geq 1 & L/2 \end{array}$$

$$c_k = \begin{cases} \frac{1}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, & k = 0 \\ \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, & k \geq 1 \end{cases}$$

do a SLICK TRICK TO CLEAN (this) UP!

Write

$$\boxed{f(x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right).}$$

Then,

$$a_0 = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, \quad k = 0$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, \quad k \geq 1$$

so

$$\boxed{a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{k\pi x}{L}\right) dx, \quad \text{ALL } k \geq 0}.$$

We call $\square + \square$ the FOURIER COSINE SERIES of $f(x)$ on $[0, L]$.

Remember $\frac{1}{2} a_0$!

Example 3 (cosines and sines)

$[-L, L]$ $\quad \left\{ \cos\left(\frac{k\pi x}{L}\right) \right\}_{k=0}^{\infty} \cup \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$

$$f \sim \sum c_\ell \varphi_\ell(x)$$

$$c_\ell = \frac{\langle f, \varphi_\ell \rangle}{\langle \varphi_\ell, \varphi_\ell \rangle}$$

eg, L for sines

Very Similar to Ex 1 & 2.

$$\boxed{\begin{aligned} f(x) &\sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 1 \\ a_k &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \qquad \text{ALL } k \geq 0 \end{aligned}}$$

FOURIER SERIES of f on $[-L, L]$.

EACH CASE is important in its own right.

MEMORIZE!

Don't worry about even/odd yet!!

Note:

$[0, L]$, $[0, L]$, $[-L, L]$.

Note:

$\frac{n\pi x}{L}$ (say this to yourself — habit "100 times")

Simplest for $L = \pi$; this is the "classical" case.

$\boxed{\text{Euler} \sim 1750, \quad \text{Fourier} \sim 1820}$

Can now do "zillions" of

explicit examples by plugging

in simple choices of f. p. 381 382

Very instructive — but may be long.

Secretly know we have equality (except finite # of x).

(A) Dust off integration techniques. EG parts! (Calc 1.)

(B) Careful attention to **L** and **format**.

(C) reasonable f? $\left\{ \begin{array}{c} \text{for our} \\ \text{purposes} \end{array} \right\}$

* Reasonable f ?

For 4567

Want $f(x)$ to be piecewise continuous.

$[a, c_1]$ $[c_1, c_2]$ $[c_2, b]$

$\uparrow$ $\uparrow$ $\uparrow$

$g_1(x)$ $g_2(x)$ $g_3(x)$

Have g_1, g_2, g_3 continuous on their respective CLOSED intervals. Take:

$$f(x) = \begin{cases} g_1(x), & a < x < c_1 \\ g_2(x), & c_1 < x < c_2 \\ g_3(x), & c_2 < x < b \end{cases}$$

Arbitrary at a, c_1, c_2, b

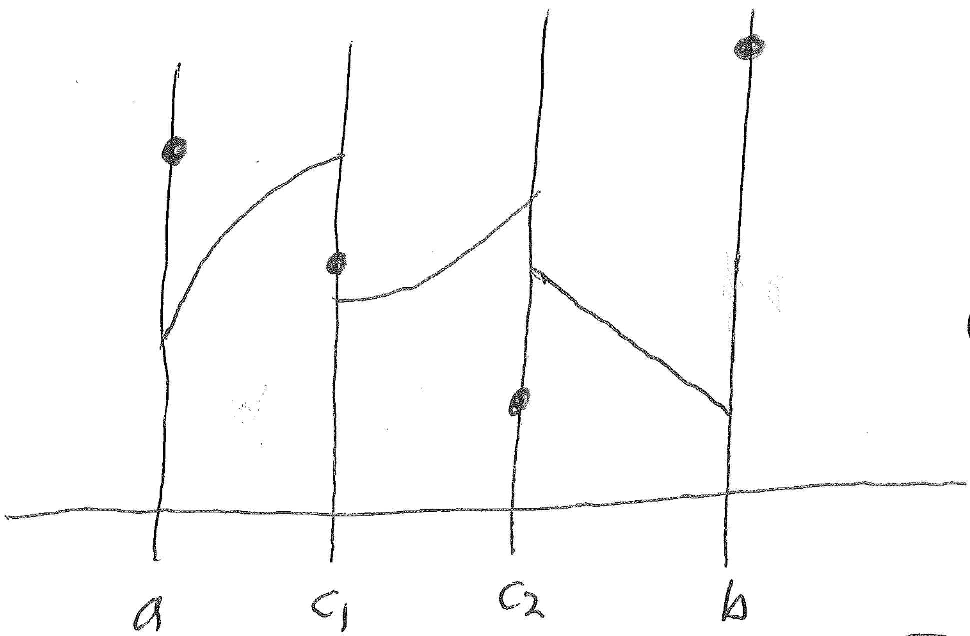
Similarly for $a < c_1 < c_2 < \cdots < c_M < b$.

Similarly for piecewise C^1, C^2, C^3, etc.

Piecewise C^1 $f(x)$.

note **Jump** Discontinuities

• arbitrary



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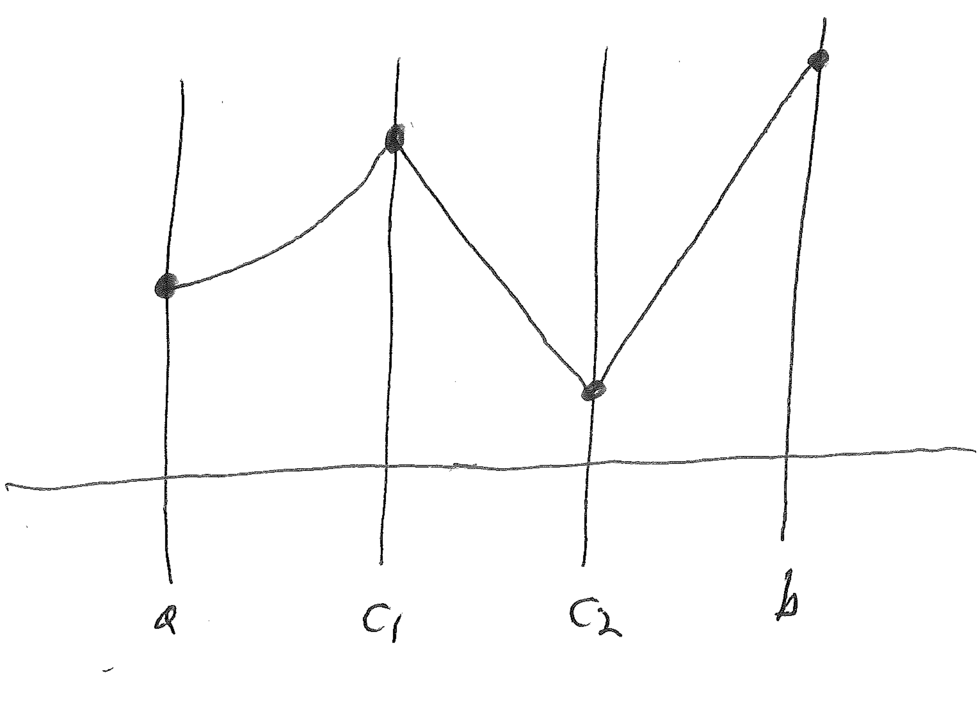
* Piecewise C^1, but continuous.



Integration Techniques.

Baby Calculus (integ by parts)

Let $u(x)$ and $v(x)$ be C^1 on $[A, B]$.

Then:

$$\int_A^B u(x)v'(x)\,dx = [u(x)v(x)]_A^B - \int_A^B v(x)u'(x)\,dx$$

$$\left\{ \int_A^B u\,dv = [uv]_A^B - \int_A^B v\,du \right\}.$$

Learn how to do parts correctly and rapidly!

Theorem (requires proof)

Let u and v be piecewise C^1 on $[A, B]$ and also continuous on $[A, B]$. Then:

$$\int_A^B u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_A^B - \int_A^B v(x)u'(x)\,dx.$$

[Not true if u or v fails to be continuous.]

(Will Discuss later.)

Example

Find **Fourier** **sine** **series** of $f = 1-x$ on $[0,1]$.

Sol.

$$1-x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$$

$$b_n = \frac{2}{1}\int_0^1 (1-x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$b_n = 2\int_0^1 (1-x)\, d\left[-\frac{\cos(n\pi x)}{n\pi}\right]$$

$$b_n = -\frac{2}{n\pi}\int_0^1 (1-x)\, d\left[\cos(n\pi x)\right]$$

$$b_n = -\frac{2}{n\pi}\left[(1-x)\cos(n\pi x)\Big]_0^1 - \int_0^1 \cos(n\pi x)(-1)dx\right]$$

$$b_n = -\frac{2}{n\pi}\left[0 - 1\cdot 1 + \int_0^1 \cos(n\pi x)\,dx\right]$$

$$b_n = -\frac{2}{n\pi}\left[-1 + \left.\frac{\sin n\pi x}{n\pi}\right]_0^1\right]$$

$$b_n = \frac{2}{n\pi} + 0 = \frac{2}{n\pi}$$

$$\Downarrow$$

$$\boxed{1 - x \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x) \quad \text{on } [0,1]}$$

N. B. Put $x = \frac{1}{2}$. Wonder:

$$\frac{1}{2} \overset{?}{=} \frac{2}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right]$$

$$\Leftrightarrow \frac{\pi}{4} \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \cdots$$

Yes!! Looks good!

$$\boxed{\begin{array}{l} \arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} \pm \cdots \\ 0 \leqq t \leqq 1 \end{array}}$$

Let's do another ~~a few more~~ example of FSS ~~and FCS~~.

Example

Find FSS of $f(x) = 1-2x$ on $[0,1]$.

by Euler

Sol.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1}\right)$$

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

Plug and chug!

$$b_n = 2\int_0^1 (1-2x)\sin(n\pi x)\,dx$$

$$= 2\int_0^1 (1-2x)\,d\left[\frac{-\cos(n\pi x)}{n\pi}\right]$$

$$= -\frac{2}{n\pi}\int_0^1 \underbrace{(1-2x)}_{u}\,d[\underbrace{\cos(n\pi x)}_{v}]$$

$$= -\frac{2}{n\pi}\Big[\ (1-2x)\cos(n\pi x)\Big]_0^1$$

$$- \int_0^1 \underbrace{\cos(n\pi x)}_{v}\ \underbrace{(-2)\,dx}_{du}\ \Big]$$

$$= -\frac{2}{n\pi}\Big[(-1)\cos(n\pi) - 1\cdot 1$$

$$+ 2\int_0^1 \cos(n\pi x)\,dx\ \Big]$$

$$= -\frac{2}{n\pi}\left[(-1)^{n+1} - 1 + 2\,\frac{\sin n\pi x}{n\pi}\Big]_0^1 \right]$$

$$= -\frac{2}{n\pi}\left[(-1)^{n+1} - 1 + 0 \right]$$

$$= -\frac{2}{n\pi}\begin{bmatrix} 0, & n \text{ odd} \\ -2, & n \text{ even} \end{bmatrix}$$

$$= \begin{bmatrix} 0, & n \text{ odd} \\ \frac{4}{n\pi}, & n \text{ even} \end{bmatrix}$$

So,

$$\boxed{1 - 2x \sim \sum_{\substack{n \text{ even} \\ n \geqq 1}} \frac{4}{n\pi} \sin(n\pi x) \quad \text{on } [0,1]}.$$

SAVE THIS!

PARTS

$$\int_A^B u(x)v'(x)\,dx = \left[u(x)v(x)\right]_A^B - \int_A^B v(x)u'(x)\,dx \,.$$

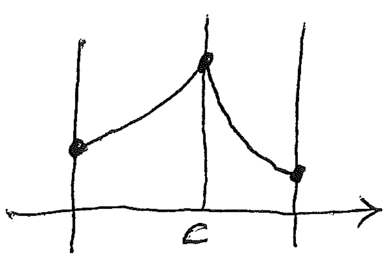
True if u & v are C^1 on $[A, B]$.

Theorem

Integ by parts is <u>also</u> true if u & v are <u>piecewise C^1</u> <u>AND</u> <u>continuous</u> on $[A, B]$.

Proof

Let's suppose that there is just one point c, $A < c < B$, where a "corner" can occur in u and v.



Remember each "strand" is a nice C^1 function g_1 ; g_2 (from last lecture).

Do parts on $[A, c]$ and $[c, B]$. OK

$$\int_A^c u(x)\,v'(x)\,dx = \underline{\underline{u(c-)v(c-)} - u(A)v(A)} - \int_A^c v(x)u'(x)\,dx$$ OK

$$\int_c^B u(x)\,v'(x)\,dx = \underline{u(B)v(B) - \underline{u(c+)v(c+)}} - \int_c^B v(x)\,u'(x)\,dx$$ OK

Add! Get:

$$\int_A^B u(x)\,v'(x)\,dx = \underline{u(B)v(B) - u(A)v(A)} + \underline{\underline{u(c-)v(c-) - u(c+)v(c+)}} - \int_A^B v(x)u'(x)\,dx$$

We see that, for integ by parts to hold,

NEED
$$u(c-)v(c-) - u(c+)v(c+) = 0.$$

Otherwise *it* **fails**! Simplest way is to insist that:

$$u(c-) = u(c+)$$
$$v(c-) = v(c+).$$

I.e., that u & v be continuous!
[then $u(c-) = u(c+) = u(c)$, etc.]

Similarly for $c_1, \ldots, c_M$.

QED