BASIC FACT

$f(x)$ given on $0 \le x \le L$.

$$f_{even}(x) = \begin{cases} f(x), & 0 < x \le L \\ f(0), & x = 0 \\ f(-x), & -L \le x < 0 \end{cases}$$

Called **THE EVEN EXTENSION** of f.

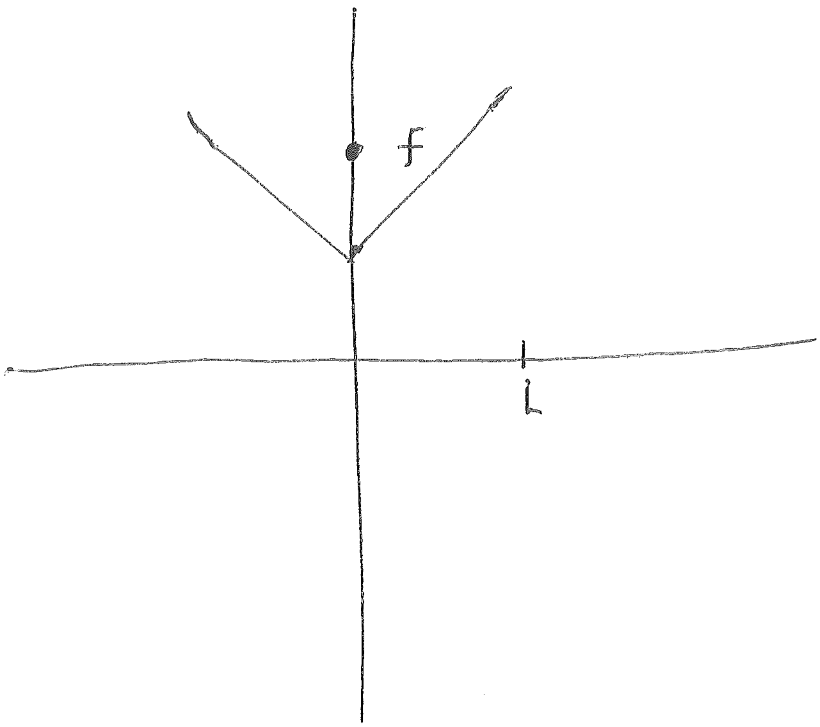
It is even.

$$f_{odd}(x) = \begin{cases} f(x), & 0 < x \le L \\ 0, & x = 0 \\ -f(-x), & -L \le x < 0 \end{cases}$$

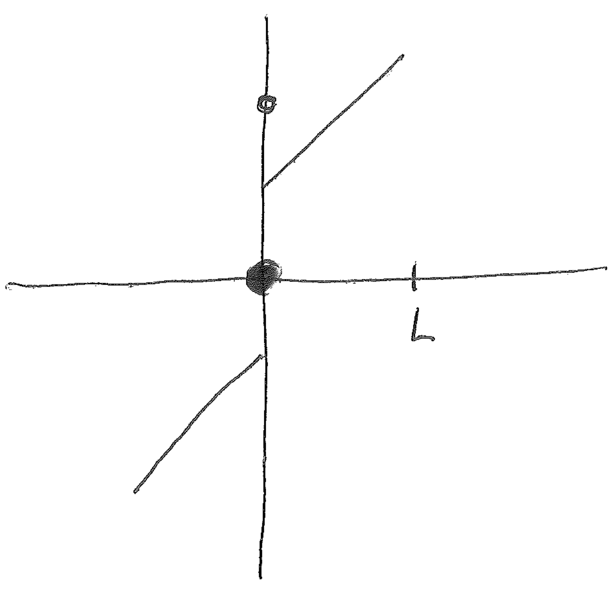
Called **THE ODD EXTENSION** of f.

It is odd.

f_{even} and f_{odd} both "extend" f.



f_{even}



f_{odd}

$f(x)$ given on $[0, L]$.

Form f_{even}, f_{odd}.

The even function, periodic $(2L)$, that extends f_{even} is called the even $(2L)$-periodic extension of f.

$$F_e(x)$$

The odd function, periodic $(2L)$, that extends f_{odd} is called the odd $(2L)$-periodic extension of f.

$$F_\theta(x)$$

$F_\theta(nL) = 0$

Ambiguities at $\pm L$?

$f(x)$ given on $[-L, L]$

(or on $[\alpha, \alpha + 2L]$)

Can form "the" $2L$-periodic extension of f.

(Monday)

$F(x)$

Note: 3 separate concepts!

Notice too:

graph of $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

is **2L**-periodic; also **odd**

---

graph of $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

is **2L**-periodic; also **even**

---

graph of

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

is **2L**-periodic

Baby Fact.

Let $g(x)$ be ω-periodic.

Then:

$$\int_x^{x+\omega} g(t)\,dt \quad \text{is } \underline{\text{independent}} \text{ of } x .$$

("Integral over a period")

Graphical proof = easy! $\blacksquare$

Or, for continuous g, take $c \approx -\infty$ and write

$$I(x) = \int_x^{x+\omega} g\,dt = \int_c^{x+\omega} g\,dt - \int_c^{x} g\,dt .$$

Apply <u>fund thm of integral calc</u>!! Get

$$I'(x) = g(x+\omega) - g(x) = 0 .$$

{at each x}

Please remember, in piecewise continuous function context, that changing the value of h at M points in $[a, b]$ does **not** change value of $\int_a^b h(x)\,dx$!

$M < \infty$

Corollary of Baby Fact

α = anything. Then,

$$\int_{\alpha}^{\alpha+2L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

$$\int_{\alpha}^{\alpha+2L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{\ell\pi x}{L}\right)dx = \begin{cases} 0, & k \neq \ell \\ L, & k = \ell \geq 1 \\ 2L, & k = \ell = 0 \end{cases}$$

$$\int_{\alpha}^{\alpha+2L} \cos\left(\frac{k\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0 .$$

VERY important!

So,

$$\left\{ \cos \frac{k\pi x}{L} \right\}_{k=0}^{\infty} \cup \left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

orthogonal on $[\alpha, \alpha + 2L]$

$f(x)$ on $[\alpha, \alpha + 2L]$ GIVEN

Can make FS of f !!

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

$$A_n = \frac{1}{L} \int_{\alpha}^{\alpha + 2L} f(x) \cos \frac{n\pi x}{L} \, dx \qquad n \geq 0$$

$$B_n = \frac{1}{L} \int_{\alpha}^{\alpha + 2L} f(x) \sin \frac{n\pi x}{L} \, dx \qquad n \geq 1$$

Example $3\frac{1}{2}$ ← (or concept)

"paraphrase of earlier one"

concept $3\frac{1}{2}$

Example (plug and chug)

Find **FS** of $f = 1 - 2x$ on $[0, 1]$.

Solo.

$a = 0 \qquad 2L = 1 \qquad \text{so} \quad L = \frac{1}{2}$

$$1 - 2x \sim \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{1/2} + B_n \sin \frac{n\pi x}{1/2} \right)$$

$$A_n = \frac{1}{1/2} \int_0^1 (1-2x) \cos \frac{n\pi x}{1/2} \, dx \qquad n \geq 0$$

$$B_n = \frac{1}{1/2} \int_0^1 (1-2x) \sin \frac{n\pi x}{1/2} \, dx \qquad n \geq 1$$

$$A_n = 2 \int_0^1 (1-2x) \cos(2n\pi x) \, dx$$

$$B_n = 2 \int_0^1 (1-2x) \sin(2n\pi x) \, dx$$

Similar to our earlier FSS example — but NOT the same thing (or concept).

Must grind it out and see!

$$A_0 = 2\int_0^1 (1-2x)\,dx = 2[1-1] = 0$$

$$A_n = 2\int_0^1 (1-2x)\cos(2n\pi x)\,dx \qquad n \geq 1$$

$$= 2\int_0^1 (1-2x)\,d\left[\frac{\sin 2n\pi x}{2n\pi}\right]$$

$$= \frac{1}{n\pi}\int_0^1 (1-2x)\,d[\sin 2n\pi x]$$

$$= \frac{1}{n\pi}\left[(1-2x)\sin 2n\pi x\Big]_0^1 - \int_0^1 \sin(2n\pi x)(-2)\,dx\right]$$

$$= \frac{1}{n\pi}\left[0 + 2\int_0^1 \sin(2n\pi x)\,dx\right]$$

$$= \frac{1}{n\pi}\left[2 \left. \frac{-\cos(2n\pi x)}{2n\pi} \right]_0^1 \right]$$

$$= \frac{1}{n\pi}\left[\frac{-2}{2n\pi} \right](1-1) = 0$$

$$\boxed{A_n = 0\,, \quad n \geq 0} \checkmark$$

$$B_n = 2\int_0^1 (1-2x)\sin(2n\pi x)\,dx$$

$$= 2\int_0^1 (1-2x)\, d\left[\frac{-\cos 2n\pi x}{2n\pi}\right]$$

$$= -\frac{1}{n\pi}\int_0^1 (1-2x)\, d\left[\cos 2n\pi x\right]$$

$$= -\frac{1}{n\pi}\left[\left.(1-2x)\cos(2n\pi x)\right]_0^1 - \int_0^1 \cos(2n\pi x)\cdot(-2)\,dx \right]$$

$$= -\frac{1}{n\pi}\left[-1 - 1 + 2\int_0^1 \cos(2n\pi x)\,dx \right]$$

$$= -\frac{1}{n\pi}\left[-2 + 2\,\frac{\sin 2n\pi x}{2n\pi}\Big]_0^1 \right]$$

$$= \frac{2}{n\pi} + 0$$

$$B_n = \frac{2}{n\pi}, \quad n \geq 1$$

So, as <u>concept $3\frac{1}{2}$</u> FS on $[0,1]$,

$$1-2x \sim \frac{1}{2}(0) + \sum_{n=1}^{\infty}\left(0 + \frac{2}{n\pi}\sin\frac{n\pi x}{1/2}\right)$$

$$1-2x \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(2n\pi x)$$

on $[0,1]$.

What was the (earlier) FSS answer?

$$1-2x \sim \sum_{\substack{n \text{ even} \\ n \geq 1}} \frac{4}{n\pi} \sin(n\pi x)$$

on [0, 1]

Let $n = 2m$, $m \geq 1$.

Right Hand Series

$$= \sum_{m=1}^{\infty} \frac{4}{2m\pi} \sin(2m\pi x)$$

$$= \sum_{m=1}^{\infty} \frac{2}{m\pi} \sin(2m\pi x)$$

It Agrees!!

Very!

LUCKY

Concept 1 vs Concept $3\frac{1}{2}$

Example

Find FCS of $f(x) = x^2$ on $[0,1]$.

Sol.

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right)$$

$$a_n = \frac{2}{1} \int_0^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx \qquad n \geq 0$$

Plug and chug!

$$a_0 = 2 \int_0^1 x^2 dx = 2 \left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$a_n = 2 \int_0^1 x^2 \cos(n\pi x)\, dx \qquad n \geq 1$$

Parts!

$$a_n = 2\int_0^1 x^2\, d\left[\frac{\sin n\pi x}{n\pi}\right]$$

$$= \frac{2}{n\pi}\int_0^1 x^2\, d\left[\sin n\pi x\right]$$

(x^2 ← u, $\sin n\pi x$ ← v)

$$= \frac{2}{n\pi}\left[x^2 \sin(n\pi x)\Big]_0^1 - \int_0^1 \sin(n\pi x)\cdot 2x\, dx\right]$$

$$= \frac{2}{n\pi}\left[0 - 0 - 2\int_0^1 x \sin(n\pi x)\, dx\right]$$

$$= -\frac{4}{n\pi}\int_0^1 x \sin(n\pi x)\, dx$$

Parts <u>again</u>!

$$= -\frac{4}{n\pi}\int_0^1 x\, d\left[\frac{-\cos(n\pi x)}{n\pi}\right]$$

$$= \frac{4}{n^2\pi^2}\int_0^1 x\, d\left[\cos(n\pi x)\right]$$

$$= \frac{4}{n^2\pi^2}\left[x\cos(n\pi x)\Big]_0^1 - \int_0^1 \cos(n\pi x)\,dx \right]$$

$$= \frac{4}{n^2\pi^2}\left[\cos(n\pi) - 0 - \frac{\sin(n\pi x)}{n\pi}\Big]_0^1 \right]$$

$$= \frac{4}{n^2\pi^2}\left[\cos(n\pi) \right]$$

Do <u>NOT</u> leave like this!!

$$\cos(n\pi) = (-1)^n$$

$$a_n = \frac{4}{n^2\pi^2}(-1)^n \,, \quad n \geq 1$$

$$x^2 \sim \frac{1}{2}\left[\frac{2}{3}\right] + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)$$

$$\boxed{x^2 \sim \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\pi x)} \quad \text{on } [0,1]$$

For fun, let's try to match this to pp. 381, 382.
✓

FCS

(381) $c = 1$ (See x^2.) Get:

$$x^2 = \frac{1}{3} + \frac{4 \cdot 1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{1}, \quad 0 < x < 1$$

It fits. ✓✓ Easy! Trivial!

---

(382) ← FSS See:

(1 − 2x NEXT) on [0,1]

1 on $0 < x < \pi$

x on $0 < x < \pi$.

Nothing else matches up.

But, wait!!

$$\left\{ \begin{array}{l} x = \frac{L}{\pi} t \\ \\ t = \frac{\pi x}{L} \end{array} \right\} \qquad 0 \leq t \leq \pi, \quad 0 \leq x \leq L$$

change-of-scale

Know: (382)

$$1 = \frac{4}{\pi} \sum_{m \text{ odd}} \frac{\sin(mt)}{m} \qquad 0 < t < \pi$$

$$t = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt) \qquad 0 < t < \pi.$$

$$L = 1; \; x = \frac{t}{\pi} \quad \text{with } 0 \leq t \leq \pi$$

So,

$$1 - 2x = 1 - \frac{2}{\pi} t \quad \text{with } 0 \leq t \leq \pi.$$

We should be OK to substitute!

$$t = \pi x$$

$t = \pi x$ $\quad 0 < x < 1$

$$\underline{1} = \frac{4}{\pi} \sum_{m \text{ odd}} \frac{\sin(m \cdot \pi x)}{m}$$

$$\pi x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n \cdot \underline{\pi x}) \Rightarrow$$

t

$$\underline{-2x} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)$$

$$\frac{4}{\pi}\frac{(-1)^n}{n}$$

Add!!

$$\therefore \quad 1-2x = \frac{4}{\pi} \sum_{n=1}^{\infty} \begin{Bmatrix} \frac{1}{n}, & n \text{ even} \ \checkmark \\ \frac{1}{n} - \frac{1}{n}, & n \text{ odd} \end{Bmatrix} \cdot \sin(n\pi x)$$

So

$$1-2x = \frac{4}{\pi} \sum_{\substack{n \text{ even} \\ n \geq 1}} \frac{1}{n} \sin(n\pi x)$$

$0 < x < 1$

So, we get by patience on 382,

$$f(x) = 1-2x = \frac{4}{\pi}\sum_{\substack{n \text{ even} \\ n \geqq 1}} \frac{1}{n}\sin(n\pi x)$$

$$0 < x < 1$$

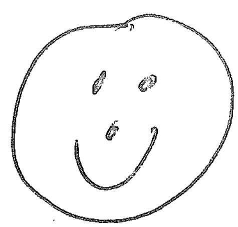
vs.

on Monday

$$1-2x \sim \sum_{\substack{n \text{ even} \\ n \geqq 1}} \frac{4}{n\pi}\sin(n\pi x).$$

OK ✓✓✓ Great!

Persistence plus change of scale



btw — $0 \lesssim x \lesssim L$, $\quad 0 \lesssim t \lesssim \pi$

$$x = \frac{L}{\pi} t$$

$$g(t) \sim \sum_{n=1}^{\infty} b_n \sin(nt) \qquad \underline{\text{given}} \text{ on } [0,\pi]$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} g(t) \sin(nt)\, dt$$

We **THINK**:

$$g\left(\frac{\pi x}{L}\right) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{on } [0, L]$$

Need:

$$\underline{B_n} = \frac{2}{L} \int_0^{L} g\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\left\{ x = \frac{L}{\pi} t, \quad \text{baby calc!} \right\}$$

$$B_n = \frac{2}{\underline{L}} \int_0^{\pi} g(t) \sin(nt) \underline{\frac{L}{\pi}}\, dt$$

Aha! Baby stuff! $\boxed{B_n = b_n}$

etc etc

2 notational issues:

"Conventions", if you will.

For Fourier series on $[-L, L]$, it is <u>customary</u> to group terms

$$g \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

[Deeper reason later.]

Next —

Any f on $[0, L]$.

Assume:

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

à la FSS.

I call **RHS** **FSS(f)**.

(right hand SERIES)

---

f on $[0, L]$ again.

Similarly **FCS(f)**. **AS A** SERIES!!

---

g on $[-L, L]$.

Similarly **FS(g)**. **AS A** SERIES!!

i.e., $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

Just Notation...

But, there is a **useful** point here.

Suppose that:

$$g = c_1 g_1 + c_2 g_2 \quad \text{on } [-L, L].$$

Then:

$$\boxed{FS(g) \equiv c_1 FS(g_1) + c_2 FS(g_2)}$$

in natural term-by-term sense!!

"linearity"

E.G.,

$$a_n(g) = \frac{1}{L}\int_{-L}^{L} g(x)\cos\frac{n\pi x}{L}\,dx$$

substitute $g = c_1 g_1 + c_2 g_2$

$$= c_1\left\{\frac{1}{L}\int_{-L}^{L} g_1(x)\cos\frac{n\pi x}{L}\,dx\right\}$$

$$+ c_2\left\{\frac{1}{L}\int_{-L}^{L} g_2(x)\cos\frac{n\pi x}{L}\,dx\right\}$$

so that

$$a_n(g) = \underline{\underline{c}}_1\, a_n(g_1) + \underline{\underline{c}}_2\, a_n(g_2) .$$

Similarly

$$b_n(g) = \underline{\underline{c}}_1\, b_n(g_1) + \underline{\underline{c}}_2\, b_n(g_2) .$$

"CHUNKS!"