EASY and IMPORTANT

Theorem

Let $g(x)$ be **even** on $[-L, L]$.
Then:

$$\underset{\text{on } [-L,L]}{FS(g)} \equiv \underset{\text{on } [0,L]}{FCS(g)} .$$

Theorem

Let $h(x)$ be **odd** on $[-L, L]$.
Then:

$$\underset{\text{on } [-L,L]}{FS(h)} \equiv \underset{\text{on } [0,L]}{FSS(h)} .$$

Just do 2^{nd} thm. $h(x)$ odd.

FS(h)

your $b_n = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi x}{L}\right)dx$

your $a_n = \frac{1}{L}\int_{-L}^{L} h(x)\cos\left(\frac{n\pi x}{L}\right)dx$

{stop and think for a second}

Yes, but $h(x)\sin\left(\frac{n\pi x}{L}\right)$ is EVEN
and $h(x)\cos\left(\frac{n\pi x}{L}\right)$ is ODD.

So,

the $b_n = \frac{2}{L}\int_{0}^{L} h(x)\sin\left(\frac{n\pi x}{L}\right)dx$

the $a_n = 0$.

Aha!

So, as an infinite series, we see

$$FS(h) = 0 + \sum_{n=1}^{\infty}\left(0\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where
$$b_n = \frac{2}{L}\int_0^L h(x)\sin\frac{n\pi x}{L}\,dx$$

but the right hand side is exactly $FSS(h)$.

on $[0, L]$

QED!

⟨Similarly for 1^{st} theorem.⟩

$FS(g) \equiv FCS(g)$ g even

$FS(h) \equiv FSS(h)$ h odd $[-L, L]$

Take absolutely any $f(x)$ on $[-L, L]$.

Write

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{g(x)} + \underbrace{\frac{f(x)-f(-x)}{2}}_{h(x)}.$$

CHEAP TRICK. Notice:

g is EVEN ; h is ODD.

So,

381   382

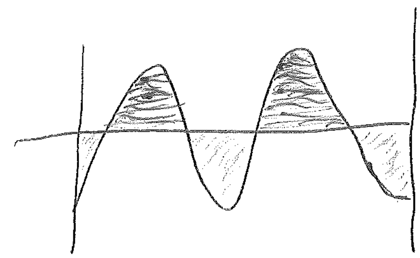
$$FS(f) = FCS(g) + FSS(h).$$

AS SERIES.

Recall:

$$\int_a^b h(x)\,dx = \text{signed area}$$

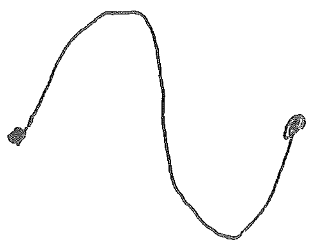
for $h(x)$ of varying sign



Now,

Chapter 2.

Smallest Period?



High School Trig!

$\sin(x)$ $\quad 2\pi$

$\cos(x)$ $\quad 2\pi$

$\sin(\alpha x)$ $\quad \dfrac{2\pi}{\alpha}$! $\qquad \alpha > 0$

etc $\qquad \sin[\alpha x + \beta]$

$\sin(10^9 x)$ $\quad \cos(10^9 x)$

oscillate ("repeat") <u>VERY</u> fast!

$$\Delta x \simeq \frac{2\pi}{10^9}$$

etc, etc

Have —

Integrals with $\boxed{\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx}$ 7

$\cos\left(\frac{n\pi x}{L}\right)$, $\sin\left(\frac{n\pi x}{L}\right)$ n large ;

also,

$\cos(\omega x)$, $\sin(\omega x)$ ω large .

EG,

a, b <u>given</u>
g given

$$\int_a^b g(x) \cos(\omega x)\, dx$$

$$\int_a^b g(x) \sin(\omega x)\, dx \quad .$$

<u>Why</u> do these approach 0

as <u>$\omega \to \infty$</u> (or $n \to \infty$) ???

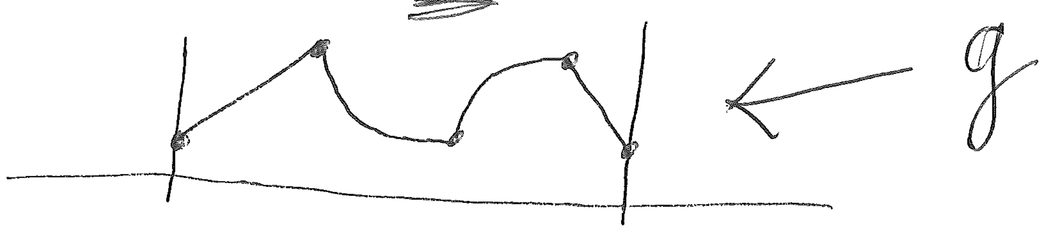
Must remember that $\cos(\omega x)$, $\sin(\omega x)$ oscillate very fast.

Think $\Delta x = \frac{2\pi}{\omega}$ and SIGNED AREA.

Rough Calculation First.

Assume that $g(x)$ is continuous on $[a, b]$. Also that it is C^1 (or at least piecewise C^1).

$$\left\{ \text{Let } |g(x)| \leqq M, \text{ say.} \right\}$$



Do integ by parts! No problem!

$$\int_a^b g(x)\cos(\omega x)\,dx \longleftarrow I(\omega) \text{ ~~say~~}$$

$$= \int_a^b g(x)\, d\left[\frac{\sin \omega x}{\omega}\right]$$

$u = g(x)$

$v = \frac{\sin \omega x}{\omega}$

$$= \frac{1}{\omega}\int_a^b g(x)\, d[\sin \omega x]$$

$$= \frac{1}{\omega}\left[\, \left[g(x)\sin(\omega x)\right]_a^b - \int_a^b \sin(\omega x)\, g'(x)\, dx \right]$$

$$= \frac{g(b)\sin(\omega b) - g(a)\sin(\omega a)}{\omega}$$

$$- \frac{1}{\omega}\int_a^b g'(x)\sin(\omega x)\, dx$$

Parts is OK

$$|I(\omega)| \leq \frac{M \cdot 1 + M \cdot 1}{\omega}$$

$$+ \frac{1}{\omega} \left| \int_a^b g'(x) \sin(\omega x)\, dx \right|$$

$$|I(\omega)| \leq \frac{2M}{\omega} + \frac{1}{\omega} \int_a^b |g'(x)|\, dx$$

$$|I(\omega)| \leq \frac{2M + V}{\omega}$$

$$\left\{ \text{where } V = \int_a^b |g'(x)|\, dx \right\}.$$

Aha!! RHS $\to 0$ as $\omega \to \infty$.

So, $I(\omega) \to 0$ at least as fast as $\dfrac{\text{constant}}{\omega}$.

Yes

Similarly for $\int_a^b g(x)\sin(\omega x)\,dx$.

Conclude:

<u>Useful Fact.</u> $\left\{\begin{array}{l}\text{Preliminary Form of} \\ \text{Riemann-Lebesgue Lemma}\end{array}\right\}$

Given <u>any</u> <u>piecewise</u> <u>C^1</u> function $g(x)$ on interval $[A, B]$. (We <u>allow</u> g to have jump discontinuities.) Then:

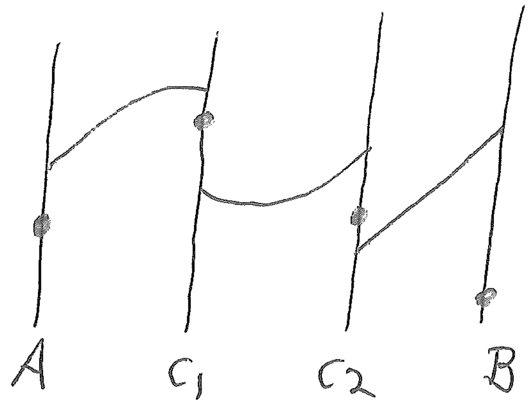
$$\left| \int_A^B g(x)\cos(\omega x)\,dx \right| \leqq \frac{\text{constant}}{\omega}$$

$$\left| \int_A^B g(x)\sin(\omega x)\,dx \right| \leqq \frac{\text{constant}}{\omega}.$$

<u>Proof</u>

Let's say $g(x)$ has 3 C^1 "pieces".

(Similarly for K pieces, $K \geqq 1$.)



$g_1 \quad [A, c_1]$

$g_2 \quad [c_1, c_2]$

$g_3 \quad [c_2, B]$.

$$\int_A^B g(x)\cos(\omega x)\,dx$$

like 8 — 10

$$= \int_A^{c_1} g_1 \cos(\omega x)\,dx \qquad \begin{array}{l} a = A \\ b = c_1 \end{array}$$

$$+ \int_{c_1}^{c_2} g_2 \cos(\omega x)\,dx \qquad \begin{array}{l} a = c_1 \\ b = c_2 \end{array}$$

$$+ \int_{c_2}^{B} g_3 \cos(\omega x)\,dx \qquad \begin{array}{l} a = c_2 \\ b = B \end{array}$$

Each <u>chunk</u> has <u>absolute</u> <u>value</u>

$$\lesssim \frac{\text{constant}}{\omega}.$$

Etc. $\blacksquare$ <u>QED</u>

Must still worry about piecewise continuous $g(x)$ such that $g'(x)$ is non-existent or "bad" in some sense!

"$x^{1/3}$"

The "Famous" Riemann-Lebesque Lemma.

VERY IMPORTANT!

Let $g(x)$ be piecewise continuous on $[A, B]$. Then:

$$\lim_{\omega \to \infty} \int_A^B g(x) \cos(\omega x)\, dx = 0$$

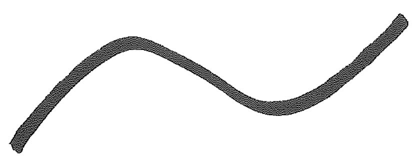
$$\lim_{\omega \to \infty} \int_A^B g(x) \sin(\omega x)\, dx = 0.$$

book p.165ff

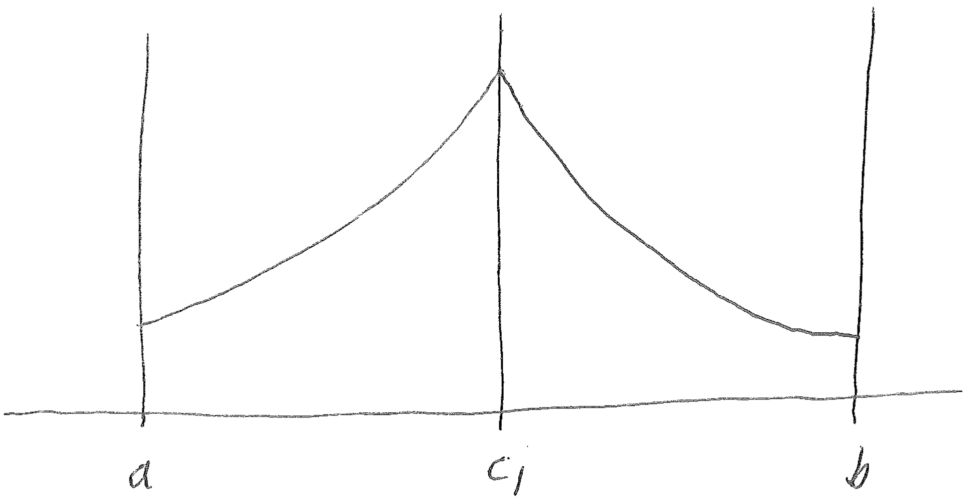
"Must drop $\frac{1}{\omega}$"

We need some preliminaries about

- UNIFORM CONTINUITY
- continuous + piecewise C^1 functions



(useful for lots of other stuff too!)



GIVEN: $f(x)$ continuous + piecewise C^1

$$f = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix}$$

Thm 1 (Fund. Thm. of Integral Calc)

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

as a piecewise continuous function

Proof

$$\int_a^c f'(x)\,dx = \int_a^c g_1'(x)\,dx = g_1(c) - g_1(a) = f(c) - f(a)$$

$$\int_c^b f'(x)\,dx = \int_c^b g_2'(x)\,dx = g_2(b) - g_2(c) = f(b) - f(c)$$

$$\Rightarrow \text{SUM} = f(b) - f(c) + f(c) - f(a) = f(b) - f(a)$$

$\boxed{\text{QED}}$

GIVEN — any
$f(x)$ defined on E.

Let $x_0 \in E$. We say $f(x)$ is continuous at x_0 when, for every $\varepsilon > 0$, we can find some $\delta > 0$ (depending on ε and x_0) so that

$|f(x) - f(x_0)| < \varepsilon$ for all $x \in E$ with $|x - x_0| < \delta$.

We say $f(x)$ is uniformly continuous on set E when, for every $\varepsilon > 0$, we can find some $h > 0$ (depending only on ε) so that

$|f(x_1) - f(x_2)| < \varepsilon$ anytime $x_j \in E$ and $|x_1 - x_2| < h$.

does not depend on position of x_1, x_2

NOT uniformly continuous:

$$f(x) = \frac{1}{x}, \quad E = \{0 < x \leq 1\}.$$

Use contradiction!

$$\left| \frac{1}{x_1} - \frac{1}{x_2} \right| < \frac{1}{100} \quad (\text{say})$$

whenever $|x_2 - x_1| < h$

Take x_1 extremely close to 0.
Take $x_2 = 2x_1$, still extremely close to 0.

Get:

$$\left| \frac{1}{x_1} - \frac{1}{2x_1} \right| < \frac{1}{100}$$

$$\Rightarrow \frac{1}{2x_1} < \frac{1}{100} \Rightarrow x_1 > 50$$

contrad!!

Thm 2 (shocking!)

Let $f(x)$ be **uniformly** continuous on set E. Let $x_0 \in E$. Then: f is automatically continuous at x_0.

Pf

Take $x_2 = x_0$. Take $\delta = h$. QED

Thm "2.9"

Let $f(x)$ be continuous **plus** piecewise C^1 on the closed interval $E = [\alpha, \beta]$. Then: f is UNIFORMLY CONTINUOUS on E.

Theorem 3 (very important)

Let $f(x)$ be continuous on the closed interval $E = [\alpha, \beta]$. Then: f is UNIFORMLY CONTINUOUS on E.

automatically

It's just mentioned in Churchill-Brown!

We call this property UNIFORM continuity because the "δ" or h does not depend on the location of x_1, only that $a \leqq x_1 \leqq b$.

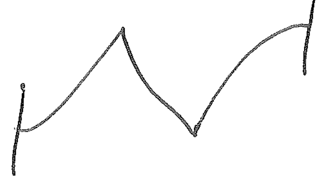
The proof of uniform continuity is usually done in a first course on "real analysis". It is an immediate consequence of what is called the Bolzano-Weierstrass property of the real numbers $\mathbb{R}$.

In effect, that real numbers are equivalent to giving an infinite decimal expansion!

VERY BASIC PROPERTY

HERE IS —

Pf of Thm "2.9"



Draw graph of $y = f(x)$ using $\{g_1, \ldots, g_m\}$.

Each g_j is $\underline{C^1}$ on a closed interval.

Find $\mathcal{M} > 0$ so that $|g_j'(x)| \leq \mathcal{M}$.

Thus $|f'(x)| \leq \mathcal{M}, \ x \in E$.

Consider $x_1 < x_2$. Have:

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)\,dx \qquad (\text{thm } 1).$$

So,

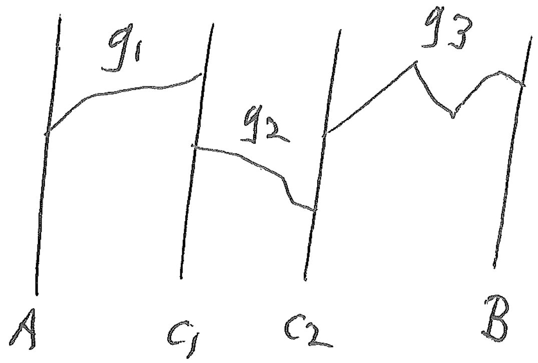
$$|f(x_2) - f(x_1)| \leq \int_{x_1}^{x_2} \mathcal{M}\,dx = \mathcal{M}(x_2 - x_1).$$

Take $h = \dfrac{\varepsilon}{2\mathcal{M}}$. This works! QED

Pf of Thm 3

Another course!! ☹

I now accept **uniform** continuity (of continuous fcns) and prove R–L lemma.



g_i = CONTINUOUS

Do, e.g., 3 "pieces". M similar.

Think! It **suffices** to prove R–L for g_1, g_2, g_3 separately.

I.e., it is **enough** to treat the case where the original g is **continuous**.

Choose any $\varepsilon > 0$. Get $\delta > 0$

so that $|g(x') - g(x'')| < \dfrac{\varepsilon}{B-A}$

whenever $|x' - x''| < \underline{\underline{\delta}}$.

UNIFORM CONTINUITY

Divide $[A, B]$ into N equal intervals, N giant. Note $\Delta x = \dfrac{B-A}{N}$.

$$A = x_0 < x_1 < x_2 < \cdots < x_N = B.$$

<u>Freeze</u> N so big that $\Delta x = \dfrac{B-A}{N} \leq \dfrac{\delta}{2}$ say.

Let

$$\ell(x) = \left\{ \begin{array}{ll} g(x_0), & x_0 \leq x < x_1 \\ g(x_1), & x_1 \leq x < x_2 \\ \vdots & \\ g(x_{N-1}), & x_{N-1} \leq x \leq \underline{\underline{x_N}} \end{array} \right\}.$$

$l(x)$ is a specific piecewise constant function. Hence, obviously piecewise C^1.

p. 11

By our preliminary form of R-L, we know:

Can also integrate it directly.

$$\lim_{\omega \to \infty} \int_A^B l(x) \cos(\omega x)\, dx = 0.$$

$$\boxed{\frac{\text{constant}}{\omega}}$$

Keep $\omega > \omega_0$ so that

$$\left| \int_A^B l(x) \cos(\omega x)\, dx \right| < \varepsilon.$$

Save this!

Notice next that

$$\int_A^B |g(x) - \ell(x)|\,dx$$

$$= \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} |g(x) - \ell(x)|\,dx$$

by p. 22

$$\leq \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\varepsilon}{B-A}\,dx$$

$$= \frac{\varepsilon}{B-A}(B-A) = \varepsilon.$$

Save this!

But, notice now that ~

$$\int_A^B g(x)\cos(\omega x)\,dx$$

$$= \int_A^B \ell(x)\cos(\omega x)\,dx$$

$$+ \int_A^B [g(x)-\ell(x)]\cos(\omega x)\,dx$$

$$\Downarrow$$

$$\left|\int_A^B g(x)\cos(\omega x)\,dx\right| \leq \left|\int_A^B \ell(x)\cos(\omega x)\,dx\right|$$

$$+ \int_A^B |g(x)-\ell(x)|\,dx$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

anytime $\underline{\underline{\omega > \omega_0}}$!

Since $\varepsilon > 0$ was arbitrary, we have shown

$$\lim_{\omega \to \infty} \int_A^B g(x) \cos(\omega x)\, dx = 0 .$$

Similarly for $\sin(\omega x)$. <u>QED</u>

$\blacksquare$

See book, p. 165 ff