

Lemma

$$z = x + iy$$

For $y > 0$, we have

$$|\operatorname{ctn}(z) + i| \leq \frac{2}{e^{2y} - 1}$$

for $y < 0$, one has

$$|\operatorname{ctn}(z) - i| \leq \frac{2}{e^{2|y|} - 1}$$

Pf.

Since $\operatorname{ctn}(\bar{w}) = \overline{(\operatorname{ctn} w)}$ for each $w \in \mathbb{C}$, it suffices to treat $y > 0$. Observe now that

$$\begin{aligned} \operatorname{ctn}(z) + i &= i \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} + i = \frac{2ie^{iz}}{e^{iz} - e^{-iz}} \\ &= \frac{2i}{1 - e^{-2iz}} = \frac{2i}{1 - e^{-2ix} e^{2y}} \end{aligned}$$

This produces

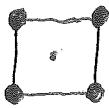
$$|\operatorname{ctn}(z) + i| = \frac{2}{|e^{2y} e^{-2ix} - 1|} \leq \frac{2}{e^{2y} - 1}$$

{Note that equality holds if, e.g., $x = 0$.} ■

Recall that $\mathbb{Z} =$ the set of all integers.

(2)

Let S_{2N+1} be the square with vertices at
 $\pm(N+\frac{1}{2}) \pm i(N+\frac{1}{2})$.



Keep $N \geq 1$. Let $r(z)$ be any rational function which tends to 0 as $z \rightarrow \infty$ at least as fast as (constant) $|z|^{-2}$ and whose poles are restricted to the set $\{0\} \cup \{\xi\}$, where ξ is some given number in $C - \mathbb{Z}$. Let $g(z)$ be a temporary abbreviation for $\pi \operatorname{ctn}(\pi z)$.

Theorem. (*) One has:

$$0 = \operatorname{Res}[r(z)g(z); z=0] + \operatorname{Res}[r(z)g(z); z=\xi] \\ + \sum_{|\xi|/k < \infty} r(k),$$

the final series being absolutely convergent.

Pf.

The infinite series involving $r(k)$ is abs convergent

(*) See REMARK on page (5) below.

(3)

because $|r(k)| \leq (\text{constant}) k^{-2}$. Observe next that $g(z)$ has a simple pole of residue 1 at each integer l . Indeed,

$$\begin{aligned} g(l+h) &= \pi \operatorname{ctn} \pi(l+h) = \pi \operatorname{ctn}(\pi l + \pi h) \\ &= \pi \operatorname{ctn}(\pi h) = \pi \frac{\cos(\pi h)}{\sin(\pi h)} \\ &= \frac{\pi [1 + O(h^2)]}{\pi h [1 + O(h^2)]} = \frac{1}{h} [1 + O(h^2)] \end{aligned}$$

by trig and standard Taylor series manipulations.
Here $|h|$ is small.

To continue, keep N much bigger than $|\xi|$.
By the Cauchy residue theorem, we have

$$\begin{aligned} (*) \quad \frac{1}{2\pi i} \oint_{\gamma_{2N+1}} r(z) g(z) dz &= \operatorname{Res}[r(z)g(z); z=0] \\ &\quad + \operatorname{Res}[r(z)g(z); z=\xi] \\ &\quad + \sum_{|z|=k \leq N} r(k) \cdot 1 \end{aligned}$$

{Think here where g has poles and where r has poles.} We propose to let $N \rightarrow \infty$ in (*).

In order to do this, note first that, by the Lemma, one has $|g(z)| \leq \pi + \pi = 2\pi$ along the horizontal portions of ∂S_{2N+1} . Along the right vertical part, one has

$$z = (N + \frac{1}{2}) + iy$$

$$\begin{aligned} \pi \operatorname{ctn} \pi \left[(N + \frac{1}{2}) + iy \right] &= \pi \operatorname{ctn} \left[\pi N + \frac{\pi}{2} + i\pi y \right] \\ &= \pi \operatorname{ctn} \left(\frac{\pi}{2} + i\pi y \right) \\ &= -\pi \tan(i\pi y) \quad \text{by trig} \\ &= -\pi i \tanh(\pi y). \end{aligned}$$

But, $\tanh(\pi y)$ increases from -1 to 1 as y ranges from $-\infty$ to $+\infty$. Accordingly: we get $|g(z)| < \pi$ along the right vertical edge of ∂S_{2N+1} . Similarly for the left edge. All told then, one has $|g(z)| \leq 2\pi$ on ∂S_{2N+1} . For large $|z|$, one also knows that $|r(z)| \leq M/|z|^2$ by hypothesis. As such:

$$\left| \oint_{\partial S_{2N+1}} r(z) g(z) dz \right| \leq \oint_{\partial S_{2N+1}} M/|z|^2 \cdot 2\pi |dz|$$



$|z| \geq N + \frac{1}{2} \text{ on } \partial S_{2N+1}$

(5)

$$\leq \frac{M}{\left(N + \frac{1}{2}\right)^2} 2\pi \cdot 4(2N+1) ,$$

which plainly tends to 0 as $N \rightarrow \infty$.

Everything is now ready for letting $N \rightarrow \infty$ in (A)
on page ③ and the Theorem follows at once.



This theorem is clearly very striking. So long as the necessary residues of $r(z)\pi \operatorname{ctn}(\pi z)$ are computable, one will be able to determine the exact value of a host of interesting infinite series!

Wow!

REMARK. It should also be noted (by reviewing the proof) that one can easily replace ξ by any finite set of values $\{\xi_1, \dots, \xi_p\} \subseteq \mathbb{C} - \mathbb{Z}$ so as to obtain a somewhat more general theorem.

This is useful.

(6)

Before turning to several examples, one readily checks by an easy long division that

$$\begin{aligned}\operatorname{ctn}(z) &= \frac{\cos(z)}{\sin(z)} = \frac{1}{z} \left[1 - \frac{1}{3}z^2 - \frac{1}{45}z^4 + O(z^6) \right] \\ &\approx \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 + O(z^5)\end{aligned}$$

for $|z|$ small. This assures us that

$$\pi \operatorname{ctn}(\pi z) = \frac{1}{z} - \frac{1}{3}\pi^2 z - \frac{1}{45}\pi^4 z^3 + O(z^5).$$

{More effort would allow us to find additional terms. Only odd powers of z would appear.}

Example 1.

Take $r(z) = \frac{1}{z^2}$. Get, by || above,

$$\operatorname{Res} \left[\frac{1}{z^2} \pi \operatorname{ctn}(\pi z); z=0 \right] = -\frac{1}{3}\pi^2$$

hence <by Thm>

$$\frac{1}{3}\pi^2 + O = \sum_{|z|=|k|<\infty} \frac{1}{k^2}.$$

(7)

In other words,

$$\boxed{\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}}.$$

Example 2.

Take $r(z) = \frac{1}{z^4}$. Get, by 1,

$$\text{Res} \left[\frac{1}{z^4} \pi \cot(\pi z); z=0 \right] = -\frac{1}{45} \pi^4$$

hence

$$\frac{1}{45} \pi^4 + 0 = \sum_{1 \leq |k| < \infty} \frac{1}{k^4}.$$

Thus,

$$\boxed{\frac{\pi^4}{90} = \sum_{k=1}^{\infty} \frac{1}{k^4}}.$$

These two "boxed" formulas are famous ones and were discovered by Euler around 1750.

Euler obtained them by way of a remarkable identity that we give here as Example 3. (Euler's proof of this identity did not use residues, since residues and the CRT did not exist yet!!) (8)

Example 3. ← a slick trick

decays like
 $|z|^{-2}$

Take $r(z) = \frac{1}{\xi-z} + \frac{1}{z} = \frac{\xi}{(\xi-z)z}$ with
 $\xi \in \mathbb{C} - \mathbb{Z}$. We need to calculate

- (a) $\text{Res}[r(z)\pi \operatorname{ctn}(\pi z); z=0]$;
(b) $\text{Res}[r(z)\pi \operatorname{ctn}(\pi z); z=\xi]$.

For (a), it is convenient to go back to page (6) and write

$$\pi \operatorname{ctn}(\pi z) = \frac{1}{z} + z H(z)$$

where $H(z)$ is a power series having $H(0) \neq 0$ and only even powers of z . For small $|z|$, we then have

$$\left(\frac{1}{z} + \frac{1}{\xi-z} \right) \left[\frac{1}{z} + z H(z) \right]$$

$$= \frac{1}{z^2} + \frac{1}{(\xi-z)z} + H(z) + \frac{\bar{z}H(z)}{\xi-z} \quad (9)$$

$$= \frac{1}{z^2} + \frac{1}{(\xi-z)z} + [\text{analytic at } 0] .$$

Hence,

$$\operatorname{Res} [r(z)\pi \operatorname{ctn}(\pi z); z=\underline{\xi}]$$

$$= 0 + \operatorname{Res} \left[\frac{1}{(\xi-z)z}; z=0 \right]$$

$$= \operatorname{Res} \left[\frac{(\xi-z)^{-1}}{z}; z=0 \right]$$

$$= \frac{1}{\xi} . //$$

This takes care of (a). For (b), near $z=\underline{\xi}$
we have:

$$\begin{aligned} \left[\frac{1}{z} + \frac{1}{\xi-z} \right] \pi \operatorname{ctn}(\pi z) &= \frac{\pi \operatorname{ctn}(\pi z)}{z} + \frac{\pi \operatorname{ctn}(\pi z)}{\xi-z} \\ &\approx [\text{analytic at } \xi] + \frac{-\pi \operatorname{ctn}(\pi z)}{z-\xi} \end{aligned}$$

hence

$$\begin{aligned} \operatorname{Res} [r(z)\pi \operatorname{ctn}(\pi z); z=\underline{\xi}] &= 0 - \pi \operatorname{ctn}(\pi \xi) \\ &= -\pi \operatorname{ctn}(\pi \xi) . // \end{aligned}$$

We can now apply the Theorem on p. ② to see that

$$0 = \frac{1}{\xi} - \pi \operatorname{ctn}(\pi\xi) + \sum_{k \neq 0} \left(\frac{1}{\xi-k} + \frac{1}{k} \right)$$

{here $k \in \mathbb{Z}$ } . Hence,

$$\boxed{\pi \operatorname{ctn}(\pi\xi) = \frac{1}{\xi} + \sum_{k \neq 0} \left(\frac{1}{\xi-k} + \frac{1}{k} \right)}$$

for each $\xi \in \mathbb{C} - \mathbb{Z}$. This identity of Euler is also famous, and is commonly called the partial fraction expansion of $\pi \operatorname{ctn}(\pi\xi)$.

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Several Notes :

- By doing assorted tricks with this "partial fraction expansion" of $\pi \operatorname{ctn}(\pi z)$, e.g. differentiating or integrating, a number of very interesting further identities can be obtained.

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- [2] Observe that the boxed formula on p. 10 can also be written as

$$\begin{aligned}\pi \operatorname{ctn}(\pi \xi) &= \frac{1}{\xi} + \sum_{l=1}^{\infty} \left(\frac{1}{\xi-l} + \frac{1}{\xi+l} \right) \\ &\approx \lim_{N \rightarrow \infty} \sum_{k=-N}^{N} \frac{1}{\xi-k}\end{aligned}$$

These alternate forms are frequently seen.

- [3] Example 1 [and its use of $\pi \operatorname{ctn}(\pi z)$] is similar in spirit to p. 254 Problem 6 in our textbook. See also p. 140 Problem 8. Incidentally: writing $T = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ and $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ immediately shows that

$$\begin{aligned}S + (-T) &= 2 \sum_{n \text{ even}} \frac{1}{n^2} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \\ &= \frac{1}{2} S\end{aligned}$$

so $T = \frac{1}{2} S$. This fits with p. 254 Problem 8.

{ On a deeper level, to connect $\frac{1}{\sin}$ with ctn , one may also want to observe that:

$$\frac{1}{\sin \pi \xi} = \frac{1}{2} \left[\operatorname{ctn} \left(\frac{\pi \xi}{2} \right) - \operatorname{ctn} \left(\frac{\pi \xi}{2} + \frac{\pi}{2} \right) \right].$$
}