What are the key Facts?

(A) Full Fourier Transform and Inversion

We define

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} \, \underline{dx} \, .$$

{ Fourier Transform }

We assume that $f(x)$ is piecewise continuous on $(-\infty, \infty)$, $f(x) \to 0$ as $x \to \pm\infty$, and $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$.

Theorem (Fourier Inversion Thm)

Suppose in the above that f is also piecewise C^1.
Then, for each x_0,

$\boxed{+i \text{ not } -i}$

$$\frac{f(x_0+) + f(x_0-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(u) e^{iux_0} \, \underline{du} \, .$$

The integral on RHS is understood to mean

$$\lim_{R\to\infty} \int_{-R}^{R} [\cdots] \, \underline{du} \, .$$

The limit as $R \to \infty$ will be approached <u>uniformly</u> on any x_0-interval, $c \leq x_0 \leq d$, over which f has <u>no</u> jumps.

(B) Fourier Cosine and Fourier Sine Transforms

Given any function $g(x)$, piecewise continuous, on $\underline{0 \leq x < \infty}$, $g(x) \to 0$ as $x \to \infty$, $\int_0^{\infty} |g(x)| \, dx < \infty$.

We define

$$\hat{g}_c(u) = \int_0^\infty g(x)\cos(ux)\,dx \qquad \{\text{Fourier cosine transf}\}$$

$$\hat{g}_s(u) = \int_0^\infty g(x)\sin(ux)\,dx \,. \qquad \{\text{Fourier sine transf}\}$$

Theorem (Fourier cosine/sine Inversion Thm)

Suppose our g is also piecewise C^1.
Then, for each $0 < x_0 < \infty$,

$$\frac{g(x_0+) + g(x_0-)}{2} = \frac{2}{\pi}\int_0^\infty \hat{g}_c(u)\cos(ux_0)\,du \,.$$

$\boxed{\text{get } g(0+) \text{ if } x_0 = 0}$

Also,

$$\frac{g(x_0+) + g(x_0-)}{2} = \frac{2}{\pi}\int_0^\infty \hat{g}_s(u)\sin(ux_0)\,du \,.$$

$\boxed{\text{get } 0 \text{ if } x_0 = 0}$

The integrals on RHS are understood to mean

$$\lim_{R\to\infty} \int_0^R [\ldots]\,du \,.$$

The limit as $R\to\infty$ will be approached uniformly on any x_0-interval, $c \le x_0 \le d$, over which g has no jumps. Here $c > 0$.
but see Ⓒ

Ⓒ Interrelations.

For g as in Ⓑ, one can always form the extensions g_{even} and g_{odd} on $-\infty < x < \infty$. THEN:

$$FT(g_{even}) = 2\,FCT(g) \;;\quad FT(g_{odd}) = -2i\,FST(g).$$

D (Other) Useful Facts to Keep In Mind for applications

Any function written as

$$Q(x) \equiv \int_0^\infty h(u)\cos(ux)\,du \qquad (x \geqq 0)$$

with nice $h(u) \to 0$ as $u \to \infty$ will satisfy

$$Q'(x) = \int_0^\infty h(u)(-u)\sin(ux)\,du\ ;$$

hence, have

$$Q'(0) = 0\ .$$

$\boxed{Q(0) \text{ will be } \int_0^\infty h(u)\,du \text{ not zero.}}$

Similarly, any function written as

$$R(x) \equiv \int_0^\infty h(u)\sin(ux)\,du \qquad (x \geqq 0)$$

with nice $h(u) \to 0$ as $u \to \infty$ will satisfy

$$R(0) = 0\ .$$

E Useful Convention

In A and B, to simplify writing, we often just write

$$f(x_0) \qquad \text{or} \qquad g(x_0),$$

when we really mean $\frac{1}{2}[f(x_0+) + f(x_0-)]$ or $\frac{1}{2}[g(x_0+) + g(x_0-)]$. Some books simply say that, AT jump discontinuities, we redefine our given function to be the average.

Ⓕ Old Version of Ⓐ ~ still used by engineers

(*) $f(x_0) = \frac{1}{\pi}\int_0^{\infty} [A(q)\cos(qx_0) + B(q)\sin(qx_0)]\,\underline{dq}$

where

(**) $A(q) = \int_{-\infty}^{\infty} f(x)\cos(qx)\,\underline{dx}$, $B(q) = \int_{-\infty}^{\infty} f(x)\sin(qx)\,\underline{dx}$.

CAUTION: note $0 \leq q < \infty$. (*) is <u>equivalent</u> to book 163(8)(9).

<u>PROOF of (*)</u> Define A and B as in (**). Take $0 < c \leq +\infty$.

For nice H,

$$\int_{-c}^{0} H(u)\,du = \int_0^c H(-w)\,dw = \int_0^c H(-u)\,du . \quad \left\{\begin{matrix}\text{Baby}\\ \text{Calculus}\end{matrix}\right\}$$

Hence, we get:

$$\int_{-R}^{R} \hat{f}(u)e^{iux_0}\,\underline{du} = \int_0^R \hat{f}(u)e^{iux_0}\,du + \int_{-R}^{0}\hat{f}(u)e^{iux_0}\,\underline{du}$$

$$= \int_0^R \hat{f}(u)e^{iux_0}\,\underline{du} + \int_0^R \hat{f}(-u)e^{-iux_0}\,\underline{du}$$

$$= \int_0^R \{\hat{f}(u)e^{iux_0} + \hat{f}(-u)e^{-iux_0}\}\,\underline{du} .$$

But,

$$\hat{f}(u)e^{iux_0} = \int_{-\infty}^{\infty} f(x)e^{-iux}\cdot e^{iux_0}\,\underline{dx} ;$$

$$\hat{f}(-u)e^{-iux_0} = \int_{-\infty}^{\infty} f(x)e^{iux}\cdot e^{-iux_0}\,\underline{dx} ;$$

$$\text{SUM} = \int_{-\infty}^{\infty} f(x)\left[e^{iu(x_0-x)} + e^{iu(x-x_0)}\right]\underline{dx}$$

$$\text{so } \{\cdots\} = 2\int_{-\infty}^{\infty} f(x)\cos u(x-x_0)\,\underline{dx} .$$

Therefore, we get

$$\frac{1}{2\pi}\int_{-R}^{R} \hat{f}(u)e^{iux_0}\,du = \frac{2}{2\pi}\int_0^R \left\{\int_{-\infty}^{\infty} f(x)\cos u(x-x_0)\,\underline{dx}\right\}\underline{du}$$

$$= \frac{1}{\pi}\int_0^R \left\{\int_{-\infty}^{\infty} f(x)[\cos ux\cdot\cos ux_0 + \sin ux\cdot\sin ux_0]\,\underline{dx}\right\}\underline{du}$$

$$= \frac{1}{\pi}\int_0^R \{A(u)\cos ux_0 + B(u)\sin ux_0\}\,\underline{du} .$$

Now let $R \to \infty$. QED by Theorem in $\boxed{A}$. ▨