3 basic ingredients:

(A) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=0$.

It suffices to prove this for $x\to 0^+$.

Need $\lim_{x\to 0^+}\frac{x-\sin x}{x\sin x}=0$. This follows from L'Hopital.

Alternatively use Taylor series. IE,

$$\frac{x-\sin x}{x\sin x}=\frac{\frac{x^3}{3!}-\frac{x^5}{5!}\pm\cdots}{x\left[x-\frac{x^3}{3!}\pm\cdots\right]}=\frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}\pm\cdots\right)}{x^2\left(1-\frac{x^2}{3!}\pm\cdots\right)}\to 0.$$

∎

(B) $\int_0^\infty \frac{\sin t}{t}\,dt=\frac{\pi}{2}\quad\left(=\lim_{A\to\infty}\int_0^A\frac{\sin t}{t}\,dt\right)$.

See book p. 164 for the proof that the A-limit exists. One uses integ by parts. Now recall book p. 32 eqs (4)~(6). We know:

$$\int_0^\pi D_N(u)\,du=\frac{\pi}{2}=\int_0^\pi\frac{\sin[(N+\frac12)u]}{2\sin(u/2)}\,du.$$

IE

$$\frac{\pi}{2}=\int_0^{\pi/2}\frac{\sin[(2N+1)v]}{\sin v}\,dv\qquad\left\{v=\frac{u}{2}\right\}.$$

The function $g(v)=\frac{1}{\sin v}-\frac{1}{v}$ is continuous on $[0,\frac{\pi}{2}]$.

By the Riemann-Lebesgue lemma [cf. (C) below] (or book p. 165)

$$\lim_{\lambda\to\infty}\int_0^{\pi/2}g(v)\sin(\lambda v)\,dv=0.$$

Write

$$\frac{\pi}{2}=\int_0^{\pi/2}\left[g(v)+\frac{1}{v}\right]\sin[(2N+1)v]\,dv\quad\text{for } N\to\infty.$$

Get:

$$\frac{\pi}{2}=0+\lim_{N\to\infty}\int_0^{\pi/2}\frac{\sin[(2N+1)v]}{v}\,dv=\lim_{N\to\infty}\int_0^{\frac{\pi}{2}(2N+1)}\frac{\sin w}{w}\,dw.$$

By taking $A=\frac{\pi}{2}(2N+1)$, we get (B). ∎

(c) Generalized Riemann-Lebesgue Lemma. Let $g(x)$ be piecewise continuous on $\mathbb{R}$ with $\int_{-\infty}^{\infty} |g(x)|\,dx < \infty$. Then:

$$\lim_{\lambda \to \infty} \int_A^B g(x) \begin{matrix} \cos(\lambda x) \\ \sin(\lambda x) \end{matrix} dx = 0$$

for any interval $[A, B]$, including cases of infinite length.

Take, e.g., $A = -\infty$, $B = +\infty$. Choose any $\varepsilon > 0$. Select $c > 0$ so big that $\int_{|x|>c} |g(x)|\,dx < \frac{\varepsilon}{2}$. Write

$$\int_{-\infty}^{\infty} g(x)\cos(\lambda x)\,dx = \int_{-c}^{c} g(x)\cos(\lambda x)\,dx + \int_{|x|>c} g(x)\cos(\lambda x)\,dx.$$

By standard R-L, know

$$\left| \int_{-c}^{c} g(x)\cos(\lambda x)\,dx \right| < \frac{\varepsilon}{2} \quad \text{for } \lambda > \lambda_\varepsilon.$$

cf. book p. 165

cos or sin

But,
$$\left| \int_{|x|>c} g(x)\cos(\lambda x)\,dx \right| \le \int_{|x|>c} |g(x)|\,dx < \frac{\varepsilon}{2} \quad \text{for ANY } \lambda.$$

So,
$$\left| \int_{-\infty}^{\infty} g(x)\cos(\lambda x)\,dx \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } \lambda > \lambda_\varepsilon. \quad \text{qed} \ \blacksquare$$

THEOREM. (Fourier inversion formula)

Let $f(x)$ be p. continuous on $\mathbb{R}$ with $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$.

Let $\hat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-iux}\,dx$, for $u \in \mathbb{R}$.

We then have:

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi} \lim_{A \to \infty} \int_{-A}^{A} \hat{f}(u) e^{iux}\,du$$

for any x where $f_R'(x)$ and $f_L'(x)$ both exist as finite numbers.

$$\boxed{\begin{array}{l} \text{Actually, we only need:} \\ \int_0^1 \left| \frac{f(x+v) - f(x+0)}{v} \right| dv < \infty, \quad \int_{-1}^{0} \left| \frac{f(x+v) - f(x-0)}{v} \right| dv < \infty \\ \text{(as improper integrals)} \end{array}}$$

Dini's condition

Proof

Take such an x. To avoid confusion, call it x_0. Note that

$$\frac{1}{2\pi}\int_{-A}^{A} \hat{f}(u)\, e^{iux_0}\, du - \frac{f(x_0+0)}{2} - \frac{f(x_0-0)}{2}$$

$$= \frac{1}{2\pi}\int_{-A}^{A}\left(\int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx\right) e^{iux_0}\, du - [\text{etc.}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-A}^{A} f(x)\, e^{iu(x_0-x)}\, du\, dx - [\text{etc.}] \quad \left(\text{by Fubini's theorem}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left[\frac{e^{iu(x_0-x)}}{i(x_0-x)}\right]_{-A}^{A} dx - [\text{etc.}]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, \frac{2i\sin[A(x_0-x)]}{i(x_0-x)}\, dx - [\text{etc.}]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\, \frac{\sin A(x_0-x)}{x_0-x}\, dx - [\text{etc.}]$$

$$= \frac{1}{\pi}\int_{x_0}^{\infty} f(x)\, \frac{\sin A(x-x_0)}{x-x_0}\, dx - \frac{f(x_0+0)}{2} + \frac{1}{\pi}\int_{-\infty}^{x_0} f(x)\, \frac{\sin A(x-x_0)}{x-x_0}\, dx - \frac{f(x_0-0)}{2}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} f(x_0+v)\, \frac{\sin Av}{v}\, dv - \frac{f(x_0+0)}{2} + \frac{1}{\pi}\int_{-\infty}^{0} f(x_0+v)\, \frac{\sin Av}{v}\, dv - \frac{f(x_0-0)}{2}$$

$$\left\{ \text{but } \int_0^{\infty} \frac{\sin Av}{v}\, dv = \frac{\pi}{2} = \int_{-\infty}^{0} \frac{\sin Av}{v}\, dv \quad \underline{\text{by (B)}} \right\}$$

Key Formula →
$$= \frac{1}{\pi}\int_0^{\infty} \frac{f(x_0+v) - f(x_0+0)}{v}\, \sin(Av)\, dv + \frac{1}{\pi}\int_{-\infty}^{0} \frac{f(x_0+v) - f(x_0-0)}{v}\, \sin(Av)\, dv$$

We need to show that both of these integrs $\underline{\underline{\to 0}}$ as $A \to \infty$. We just do the first one. Write it as:

$$\frac{1}{\pi}\int_0^{1} \frac{f(x_0+v) - f(x_0+0)}{v}\, \sin(Av)\, dv + \frac{1}{\pi}\int_1^{\infty} \frac{f(x_0+v)}{v}\, \sin(Av)\, dv$$
$$- \frac{1}{\pi} f(x_0+0) \int_1^{\infty} \frac{1}{v}\, \sin(Av)\, dv.$$

Let $g(v) = \dfrac{f(x_0+v) - f(x_0+0)}{v}$ for $0 < v \leqq 1$. For $v = 0$, define $g(0) = f_R'(x_0)$. This function g is piecewise continuous on $[0,1]$. By standard R-L lemmas

$$(\star) \qquad \lim_{A\to\infty} \int_0^1 g(v)\sin(Av)\,dv = 0 .$$

Next, let $h(v) = \dfrac{f(x_0+v)}{v}$ for $v \geqq 1$ and 0 otherwise. This function h is p. continuous on $\mathbb{R}$. Moreover,

$$\int_{-\infty}^{\infty} |h(v)|\,dv = \int_1^{\infty} \frac{|f(x_0+v)|}{v}\,dv \leqq \int_1^{\infty} |f(x_0+v)|\,dv \qquad \{w = x_0+v\}$$

$$\leqq \int_{-\infty}^{\infty} |f(w)|\,dw < \infty .$$

By (C), we then get:

$$\lim_{A\to\infty} \int_1^{\infty} h(v)\sin(Av)\,dv = 0 .$$

Note how we have an infinitely long interval.

Finally,

$$\int_1^{\infty} \frac{\sin(Av)}{v}\,dv = \int_A^{\infty} \frac{\sin(w)}{w}\,dw . \qquad \{w = Av\}$$

Therefore:

$$\lim_{A\to\infty} \int_1^{\infty} \frac{\sin(Av)}{v}\,dv = \lim_{A\to\infty} \int_A^{\infty} \frac{\sin w}{w}\,dw = 0 .$$

QED ■

in box

To get the "actually" on page 2, one needs to deduce $(\star)$ using a very slight (and easy) generalization of the standard R-L lemma. The idea for this is similar to the "trick" with $|x| > c$ that we exploited in the proof of (C).

Just write $\int_0^1 = \int_0^\delta + \int_\delta^1$ and keep δ appropriately tiny.

— PART TWO —

Let $f(x)$ be given only on $[0, \infty)$. Assume that it is piecewise continuous and satisfies

$$\int_0^\infty |f(x)|\,dx < \infty .$$

One DEFINES

$$\hat{f}_c(u) = \int_0^\infty f(x)\cos(ux)\,dx$$

Fourier cosine transform

$$\hat{f}_s(u) = \int_0^\infty f(x)\sin(ux)\,dx .$$

sine transform

Conceptually (in the back of one's mind), one thinks of f being extended as an even function in case "c", and as an odd function in case "s".

By applying THEOREM on p. 2 to these extended functions, one easily gets the following fact.

COROLLARY.

Suppose that $x > 0$ is a point where $f_R'(x)$ and $f_L'(x)$ both exist. Then:

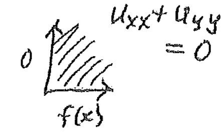
$$\frac{f(x+0)+f(x-0)}{2} = \frac{2}{\pi}\int_0^\infty \hat{f}_c(u)\cos(ux)\,du$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{2}{\pi}\int_0^\infty \hat{f}_s(u)\sin(ux)\,du .$$

If $f_R'(0)$ exists, the "cosine formula" gives $f(0+)$ at $x=0$.
[The "sine formula" gives 0.]

Example: Let $D = \{x>0, y>0\}$. Find a nice <u>bounded</u> harmonic function $u(x,y)$ on D such that

$$u(0,y) = 0, \quad u(x,0) = \text{given } f(x). \rightsquigarrow$$



<u>Solution</u> Compare book, 177–179.

For each $y>0$, $u(x,y)$ is nice on $0 \leq x < \infty$ and has value $\underline{\underline{0}}$ at $x=0$. It makes sense to write

$$u(x,y) = \frac{2}{\pi}\int_0^\infty \mathcal{U}(p,y)\sin(px)\,dp, \leftarrow \boxed{\text{we assume } u \text{ is nice}}$$

where $\mathcal{U}(p,y)$ corresponds to the FST of $u(x,y)$. (for <u>EACH</u> y)

$$u_{xx} = \frac{2}{\pi}\int_0^\infty \mathcal{U}(p,y)(-p^2)\sin(px)\,dp$$

$$u_{yy} = \frac{2}{\pi}\int_0^\infty \mathcal{U}_{yy}\sin(px)\,dp$$

$$0 = u_{xx}+u_{yy} = \frac{2}{\pi}\int_0^\infty [\mathcal{U}_{yy} - p^2\mathcal{U}]\sin(px)\,dp.$$

Need $\mathcal{U}_{yy} - p^2\mathcal{U} = 0$ for each p. Hence:

$$\mathcal{U}(p,y) = c_1(p)e^{py} + c_2(p)e^{-py}. \qquad (p>0)(y>0)$$

Presumably, for a nice bounded u, want $c_1(p) = 0$. We simply <u>assume</u> that, and then proceed:

$$\mathcal{U}(p,y) = c_2(p)e^{-py}.$$

So:

$$u(x,y) = \frac{2}{\pi}\int_0^\infty c_2(p)e^{-py}\sin(px)\,dp.$$

This is the analog of our old <u>ansatz</u> idea.

Let $y=0$. Want

$$f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(p)\sin(px)\,dp = \frac{2}{\pi}\int_0^\infty c_2(p)\sin(px)\,dp.$$

So, take $c_2(p) = \hat{f}_s(p)$.

<u>ANSWER</u> $u(x,y) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(p)e^{-py}\sin(px)\,dp$

{ We assume our original problem can have <u>AT</u> <u>MOST</u> <u>one</u> solution. This is, in fact, a theorem (since u is bounded). }

Complex Number Phobia?

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(2013)

$$f(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(u) e^{iux_0} du$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(u) e^{iux_0} du + \frac{1}{2\pi}\int_{-\infty}^{0} \hat{f}(u) e^{iux_0} du$$

$\{u = -v\}$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(u) e^{iux_0} du + \frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(-v) e^{-ivx_0} dv$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(u) e^{iux_0} du + \frac{1}{2\pi}\int_{0}^{\infty} \hat{f}(-u) e^{-iux_0} du$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{-iux} dx\right] e^{iux_0} du + \frac{1}{2\pi}\int_{0}^{\infty} \left[\int_{-\infty}^{\infty} f(x) e^{iux} dx\right] e^{-iux_0} du$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x) e^{-iu(x-x_0)} dx\,du + \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x) e^{iu(x-x_0)} dx\,du$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x) \left[e^{-iu(x-x_0)} + e^{iu(x-x_0)}\right] dx\,du$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(x)\, 2\cos u(x-x_0)\, dx\,du$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \left(\int_{-\infty}^{\infty} f(x) \cos\alpha(x-x_0)\, dx\right) d\alpha$$

THIS IS BOOK p. 163 (7)!

NOTE too that steps can be reversed

$$z = a + bi$$

$$\bar{z} = a - bi$$

USEFUL:

$$\overline{(z + w)} = \bar{z} + \bar{w}$$

$$\overline{(z \cdot w)} = \bar{z}\,\bar{w}$$

$$z\bar{z} = |z|^2 = a^2 + b^2$$

$$\overline{e^{i\theta}} = e^{-i\theta}$$

$$\overline{\sum_{j=1}^{N} c_j z_j} = \sum_{j=1}^{N} \bar{c}_j \bar{z}_j$$

$$\overline{\int_a^b f(t)\,dt} = \int_a^b \overline{f(t)}\,dt \quad \left(\begin{array}{c} a, b \\ \text{real} \\ \text{numbers} \end{array}\right)$$

$$\frac{d}{dt}\left(e^{ct}\right) = c e^{ct}, \quad c \text{ complex}$$

$$e^{(a+i\theta)t} \equiv e^{at} e^{i\theta t}$$