4567 Hejhal 4/2017

(I)

Some connections with elementary physics vis à vis Sturm – Liouville.

Recall our analysis of heat flow using Fourier's empirical flux law. Book p. 63. Specialize to case of a rod going from $x=0$ to $x=L$, with fixed (tiny) cross-sectional area A.

We got:

specific heat, density (3-d) — thermal conductivity (for unit area)

$$A\int_{x_1}^{x_2} s(x)\rho(x)\frac{\partial u}{\partial t}\,dx = \int_{x_1}^{x_2}\frac{\partial}{\partial x}\left[A\,K(x)\,u_x\right]dx$$

$$+\int_{x_1}^{x_2} g(x,t,u)\,dx$$

where $g(x,t,u)$ = the internal heat source term.

Thus, by letting $x_2 \to x_1$,

$$s(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(x)\,u_x\right] + \frac{g(x,t,u)}{A}.$$

Changing notation, get

$$\frac{\partial u}{\partial t} = \frac{1}{s(x)\rho(x)}\frac{\partial}{\partial x}\left[K(x)\,u_x\right] + Q(x,t,u).$$

See book p. 65 top (5).

In simple situations, one takes

$$Q(x,t,u) = q_0(x,t)u + q_1(x,t).$$

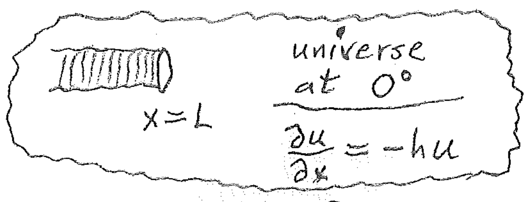
Thus:

$$u_t = \frac{1}{s(x)\,\rho(x)} \frac{\partial}{\partial x}[k(x)u_x] + q_0(x,t)u + q_1(x,t).$$

This is a general linear nonhomogeneous heat eq.

Before continuing, recall too the Newton cooling condition with respect to ambient background temperature $T = 0$. This has form [p. 74 (3)]

$$\frac{\partial u}{\partial \vec{n}} + hu = 0,$$



where $h > 0$ and unit normal $\vec{n}$ points away from the rod. [p. 74 example 1]

The insulated condition $\frac{\partial u}{\partial \vec{n}} = 0$ is also familiar, as is $u = 0$.

no flux

We are especially interested in situations amenable to separation of variables.

Clearly, then, we'll need $q_1(x,t) \equiv 0$. Also, for simplicity, we better take $\boxed{q_0(x,t) = q_0(x).}$

The boundary condition at $x = L$ should be

$$u(L,t) = 0 \quad \text{or} \quad u_x(L,t) = 0$$

$$\text{or} \quad u_x(L,t) + h_2\, u(L,t) = 0 \quad (h_2 > 0).$$

At $x = 0$, similarly

$$u(0,t) = 0 \quad \text{or} \quad u_x(0,t) = 0$$

$$\text{or} \quad -u_x(0,t) + h_1\, u(0,t) = 0 \quad (h_1 > 0).$$

With such boundary conditions, our separated "baby" solutions $A(t)\,B(x)$ must satisfy

$$A'(t)B(x) = \frac{1}{s(x)\rho(x)} \frac{\partial}{\partial x}\left[K(x)A(t)B'(x)\right] + q_0(x)A(t)B(x)$$

by p. 2

$$\Downarrow$$

$$\frac{A'(t)}{A(t)} = \frac{1}{s(x)\rho(x)} \frac{\partial}{\partial x}[K(x)B'(x)] \cdot \frac{1}{B(x)} + q_0(x) = C = -\lambda .$$

Notice the B-portion! **Get:**

$$\frac{1}{s(x)\rho(x)} \frac{1}{B(x)} \frac{d}{dx}\left[K(x)\frac{dB}{dx}\right] + q_0(x) = -\lambda$$

$$\frac{d}{dx}\left[K(x)\frac{dB}{dx}\right] + q_0(x)\, s(x)\rho(x)\, B(x) = -\lambda\, s(x)\rho(x)\, B(x)$$

OR

$$\frac{d}{dx}\left[K(x)\frac{dB}{dx}\right] + q(x)\, B(x) + \lambda \boxed{s(x)\rho(x)}\, B(x) = 0$$

where $q(x) \equiv q_0(x)\, s(x)\rho(x)$.

Here $B(x)$ satisfies the appropriate homogeneous boundary condition at $x=0$ and $x=L$.

Note how we get [book p. 210 (3)] a S-L ODE

$$(rB')' + qB + \lambda p(x) B = 0 ,$$

with $r = K(x)$, $q = q$, $p(x) = s(x)\rho(x)$.

positive positive

Sturm-Liouville Matters

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Basic Technical Tool:

$$\int_a^b (FG' + F'G)\,dx = [FG]_a^b \quad . \qquad \left(\begin{matrix}\text{by} \\ \text{calculus}\end{matrix}\right)$$

Consider the (S-L) diff eq.

(★) $\quad (ru')' + qu + \lambda p u = 0\,, \quad a \leq x \leq b \qquad (u \not\equiv 0)$

wherein $r > 0$, $p > 0$ and $r, q, p \in C^2[a,b]$. Equip (★) with one of two kinds of boundary conditions:

(A) (separated) $\quad a_1 u(a) + a_2 u'(a) = 0$ and $b_1 u(b) + b_2 u'(b) = 0$
with $(a_1, a_2) \neq (0,0)$, $(b_1, b_2) \neq (0,0)$;

<u>or</u>

(B) (periodic) when p, q, r are C^2 <u>and $(b-a)$-periodic</u> on $\mathbb{R}$,
REQUIRING $u(a) = u(b)$, $u'(a) = u'(b)$.

We say that we (then) have a <u>Sturm-Liouville-type eigenvalue problem</u>. (λ is known as the eigenvalue.)

In many cases, by a change of variable in x, <u>and</u> also in u, it is possible to transform the given S-L eigenvalue problem (★) to a <u>simpler</u> one of format

(★★) $\quad \dfrac{d^2 w}{dt^2} + Q(t) w + \lambda w = 0\,, \qquad 0 \leq t \leq L$.

Thus, in a sense, $R(t) \equiv 1$ and $P(t) \equiv 1$ here.

$$t = \int \sqrt{\frac{p(x)}{r(x)}}\,dx$$

{ Cf. Courant-Hilbert, Methods of Math. Physics, vol. I, sec. V-3-3. }

There are two BIG theorems.

THEOREM I. Given (✱) with a separated boundary condition. There will then exist infinitely many eigenvalues λ_n. These can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$; we will have $\lambda_n \to +\infty$. For each λ_n, there is only one corresponding solution u_n of (✱) up to multiplication by a constant. The functions u_n are orthogonal with respect to the (weighted) inner product

$$\langle f, g \rangle_p \equiv \int_a^b f(x) g(x) \underline{\underline{p(x)}} \, dx .$$

For any C^2 function $F(x)$ on $[a,b]$ satisfying the given separated boundary condition, we have

$$F(x) = \sum_{n=1}^{\infty} c_n u_n(x) \qquad \text{on} \quad a \leq x \leq b$$

wherein

$$c_n = \frac{\langle F, u_n \rangle_p}{\langle u_n, u_n \rangle_p} .$$

The infinite series will converge both absolutely and uniformly on $[a,b]$. Parseval's equation will hold for any piecewise continuous $G(x)$ on $[a,b]$; i.e.

$$G(x) \sim \sum_{n=1}^{\infty} d_n u_n(x) \Rightarrow$$

$$\langle G, G \rangle_p = \sum_{n=1}^{\infty} d_n^2 \langle u_n, u_n \rangle_p .$$

A similar theorem holds in the case of periodic boundary conditions except that, for each λ_n, one can have either one eigenfunction u_n or else 2 linearly independent ones, say $u_{n1}(x)$ and $u_{n2}(x)$. Things can always be arranged so that

$$\langle u_{n1}, u_{n2} \rangle_p = 0 .$$

THEOREM II. Given the situation of theorem I. Suppose that [in case (A)]

$$g(x) \leqq 0 , \quad a_1 a_2 \leqq 0 , \quad b_1 b_2 \geqq 0 .$$

Then, automatically; all $\lambda_n \geqq 0$. The same conclusion holds in case (B) anytime $g(x) \leqq 0$.

The complete proof of theorem I is rather complicated. See, e.g., the above mentioned book of Courant-Hilbert. V-3-3, V-14-X

Here's the proof that $\langle u_n, u_m \rangle_p = 0$ for $n \neq m$ in case (A). Suppose, for instance, that

$$u'(a) = a_3 u(a) , \quad u'(b) = b_3 u(b) .$$

Apply basic technical tool. Get:

$$\int_a^b [(ru_n')u_m' + (ru_n')' u_m] dx = [(ru_n')u_m]_a^b$$

$$\int_a^b [(ru_m')u_n' + (ru_m')' u_n] dx = [(ru_m')u_n]_a^b$$

$$\Downarrow$$

$$\int_a^b [(ru_n')' u_m - (ru_m')' u_n] dx = [ru_n' u_m - ru_m' u_n]_a^b .$$

WRONSKIAN

NOTE that: $u_n'(a) u_m(a) - u_m'(a) u_n(a) = a_3 [u_n(a) u_m(a) - u_m(a) u_n(a)] = 0 .$

Similarly at $x = b$. So: $[ru_n' u_m - ru_m' u_n]_a^b = 0 !$

Get:

$$\int_a^b [(ru_n')' u_m - (ru_m')' u_n]\, dx = 0 \quad ; \quad \text{i.e., by } (\star),$$

$$\int_a^b [(-qu_n - \lambda_n p u_n) u_m - (-qu_m - \lambda_m p u_m) u_n]\, dx = 0$$

$$\Downarrow$$

$$(\lambda_m - \lambda_n) \int_a^b u_m u_n \underline{p(x)}\, dx = 0 \qquad (\lambda_n \neq \lambda_m)$$

$$\Downarrow$$

$$\langle u_m, u_n \rangle_p = 0 . \qquad \text{OK}$$

Similarly in all other cases.

To check theorem $\underline{\text{II}}$ in the preceding "a_3, b_3" case is now easy. Just go back and take $\underline{u} = \underline{m}$.

Get:

$$\int_a^b [(ru_n')u_n' + (ru_n')' u_n]\, dx = [(ru_n')u_n]_a^b .$$

Plug in to get:

$$\int_a^b [r(u_n')^2 + (-qu_n - \lambda p u_n) u_n]\, dx = r(b) b_3 u_n(b)^2 - r(a) a_3 u_n(a)^2$$

$$\Downarrow$$

cf. OUR BOOK, 221 eq (4)

$$\lambda \int_a^b u_n(x)^2 p(x)\, dx = r(a) a_3 u_n(a)^2 - r(b) b_3 u_n(b)^2 + \int_a^b r(x)(u_n')^2 dx + \int_a^b (-q) u_n^2\, dx .$$

But $a_1 a_2 \leqq 0$ means $a_3 \geqq 0$. $\left(a_3 = -\frac{a_1}{a_2}\right)$

And $b_1 b_2 \geqq 0$ means $b_3 \leqq 0$. $\left(b_3 = -\frac{b_1}{b_2}\right)$

We also have $r(a) > 0$, $r(b) > 0$, $\int_a^b u_n^2 p\, dx > 0$, $\int_a^b r(u_n')^2 dx \geqq 0$, $-q \geqq 0$. $\Rightarrow$ OK

[SIMILARLY IN ALL OTHER CASES.]

II

Suppose now that one needs to solve the original NON-HOMOGENEOUS HEAT EQ

see 2 line 4

$$s(x)\rho(x)\,u_t = \frac{\partial}{\partial x}\left[K(x)u_x\right] + Q(x)u + R(x,t)$$

+ B.C.

3 line 4; put $q = Q = s(x)\rho(x)q_0(x)$

What would we do?

ANSWER

Use variation of parameters where our "basis functions" $A_n(x)$ are adapted to the given B.C. Thus, we put

Note that these corresponded to $B(x)$ at 4 line 8!

$$u(x,t) = \sum_{n=1}^{\infty} I_n(t)\,A_n(x)$$

and seek to get ODE for $I_n(t)$.

In doing this, you must first also expand R as

$$\sum_{n=1}^{\infty} R_n(t)\,A_n(x).$$

the "basis" functions

III) Full-fledged example "on the fly".

p. 248 problem 4

Solve:

$$u_t = k u_{xx} + q_0 \qquad 0 < x < 1, \ t > 0$$

$$u_x(0,t) = 0, \quad u_x(1,t) + h u(1,t) = 0 \qquad \begin{pmatrix} q_0 > 0 \\ h > 0 \end{pmatrix}$$

$$u(x,0) = 0.$$

$\boxed{u \text{ bounded}}$

Solution

Don't panic!!

p. 9

First, look at homog heat eq to get S-L functions $A_n(x)$.

$$u_t = k u_{xx}, \quad u = A(x)B(t) \qquad \text{Want: } \begin{array}{l} A'(0) = 0 \\ A'(1) + hA(1) = 0. \end{array}$$

$$\Rightarrow \quad A(x)B'(t) = k A''(x) B(t)$$

$$\frac{1}{k}\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)} = -\lambda$$

$$\Downarrow$$

want $\quad A''(x) + \lambda A(x) = 0$ on $[0,1]$ $\quad$ OK
$\qquad A'(0) = 0 \quad A'(1) + hA(1) = 0.$

Use p. 383 TABLE #1. (Or p. 221 example 1.)

$$\lambda_n = R_n^2 \quad , \quad \tan R_n = \frac{h}{R_n} \qquad (R_n > 0)$$

$$\phi_n = \sqrt{\frac{2h}{h + \sin^2 R_n}} \cos R_n x \quad , \quad \phi_n = k_n A_n(x)$$

$A_n(x)$

$$1 = \langle \phi_n, \phi_n \rangle$$
$$1 = k_n^2 \langle A_n, A_n \rangle$$
$$\Downarrow$$
$$\frac{1}{\langle A_n, A_n \rangle} = k_n^2 = \frac{2h}{h + \sin^2 R_n}$$

the "trick"

We are now ready for <u>var</u> <u>of</u> <u>parameters</u>:

$$u(x,t) = \sum_1^\infty I_n(t) \cos R_n x$$

$$\left\{ \begin{array}{l} \text{know} \\ u(x,0) = 0 \\ \text{so } I_n(0) = 0 \end{array} \right\}$$

Want:

$$u_t - k u_{xx} = g_0$$

$$\Downarrow$$

$$\sum_1^\infty I_n'(t) \cos R_n x + \sum_1^\infty k I_n(t) R_n^2 \cos(R_n x) = g_0$$

$$\sum_1^\infty [I_n'(t) + k R_n^2 I_n(t)] \cos(R_n x) = g_0 .$$

Must also expand g_0 !!

$$q_0 = \sum_1^\infty c_n \cos(R_n x) = \sum_1^\infty c_n A_n(x)$$

$$c_n = \frac{\langle q_0, A_n \rangle}{\langle A_n, A_n \rangle} = \frac{2h}{h+\sin^2 R_n} \int_0^1 q_0 \cos(R_n x)\,dx$$

$$c_n = \frac{2h}{h+\sin^2 R_n}\, q_0 \,\frac{\sin R_n}{R_n}.$$

Must now solve:

$$I_n'(t) + k R_n^2 I_n(t) = c_n \;, \quad I_n(0) = 0.$$

Use undetermined coefficients.

$$I_{part} = \frac{c_n}{k R_n^2}$$

$$I_{homog} = C e^{-k R_n^2 t}$$

$$I_n = \frac{c_n}{k R_n^2} + C e^{-k R_n^2 t}$$

$$\Rightarrow \quad I_n(t) = \frac{c_n}{k R_n^2}\left(1 - e^{-k R_n^2 t}\right).$$

So,

as above

$$u(x,t) = \sum_1^\infty \frac{c_n}{k R_n^2}\left(1 - e^{-k R_n^2 t}\right) \cos(R_n x).$$

We are effectively done!

But, we want to simplify.

OK as final answer

$$\boxed{u = \sum_{1}^{\infty} \frac{1}{kR_n^2} \frac{2h}{h+\sin^2 R_n} q_0 \frac{\sin R_n}{R_n} \left(1 - e^{-kR_n^2 t}\right) \cos(R_n x)}$$

The book likes to pull out the "steady-state" (or limiting solution) from the decaying part.

$$u = \sum_{1}^{\infty} \frac{1}{kR_n^3} (q_0 \sin R_n) \frac{2h}{h+\sin^2 R_n} \cos(R_n x)$$

$$- \sum_{1}^{\infty} \frac{1}{kR_n^3} (q_0 \sin R_n) \frac{2h}{h+\sin^2 R_n} e^{-kR_n^2 t} \cos(R_n x)$$

The first sum, call it $\Phi(x)$, simplifies!!!

Notice that: $R_n > 0$

$$\Phi(x) = \lim_{t \to \infty} u(x,t).$$

The "steady state".

We must therefore have:

$$\Phi'(0) = 0, \quad \Phi'(1) + h\Phi(1) = 0$$

$$0 = k\Phi_{xx} + q_0$$

(see p. 10).

Very Similar Idea as on page 246

Get:

$$\Phi'' = -\frac{q_0}{k}$$

$$\Rightarrow \Phi = -\frac{q_0}{2k}x^2 + C_1 x + C_2$$

substitute $\Phi'(0) = 0$, $\Phi'(1) + h\Phi(1) = 0$

$\Rightarrow$ get

$$\Phi(x) = -\frac{q_0}{2k}x^2 + \frac{q_0}{kh}\left(1 + \frac{h}{2}\right).$$

So,

$$u = -\frac{q_0}{2k}x^2 + \frac{q_0}{kh}\left(1 + \frac{h}{2}\right) - \sum_1^\infty \frac{1}{kR_n^3}(q_0 \sin R_n)\frac{2h}{h + \sin^2 R_n} e^{-kR_n^2 t}\cos(R_n x)$$

$$\boxed{u = \frac{q_0}{2k}\left[-x^2 + \frac{2}{h} + 1 - \sum_1^\infty \frac{1}{R_n^3}(\sin R_n)\frac{4h}{h + \sin^2 R_n} e^{-kR_n^2 t}\cos(R_n x)\right]}$$

exactly as in book.

Done!

$\left\{ \text{Either this } \underline{\text{or}} \ \boxed{13} \text{ top is OK as } u. \right\}$