

Proof of Uniform Continuity

Assuming The Bolzano - Weierstrass Property of \mathbb{R}

The B-W property of \mathbb{R} states that every infinite set of points E lying within a bounded closed interval $[A, B]$ necessarily has at least one accumulation point, and this lies in $[A, B]$.

See textbook p. 255 (prob 11). (*)

think
 $R = [A, B]$

THEOREM

Every continuous function $f(x)$ on a bounded closed interval $[A, B]$ is automatically uniformly continuous on $[A, B]$. I.E., for every $\epsilon > 0$, we can find a $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ anytime $|x - y| < \delta$ no matter where $\{x, y\}$ are situated in $[A, B]$.

(*) The B-W property is very basic and is proved in many books. It can be easily gotten by combining the nested intervals thm with the "bisection idea" used in my earlier 5583 handout (3 Beautiful Proofs). See pp. 2 \square , 5-7; seek intervals $R^{(k)}$ containing infinitely many points of E . The proof immediately adapts to sets E and "boxes" in \mathbb{R}^N .

P.F.

We use contradiction. If f were not uniformly continuous on $[A, B]$, there must exist some small ϵ_0 which is "bad" in the sense that, for each $n \geq 1$, we can "cook up" points $\{x_n, y_n\} \subseteq [A, B]$ so that

$$|f(x_n) - f(y_n)| \geq \epsilon_0 \quad \text{but} \quad |x_n - y_n| \leq \frac{B-A}{2^n}.$$

Look at the sequence of numbers $\{x_n\}_{n=1}^{\infty}$ starting at x_1 . (The x_n may not be distinct.) Go thru the sequence successively flagging* each entry which is a numerical repeat of an earlier one. Let $n_1 < n_2 < \dots$ be the list of indices corresponding to "survivors" (i.e., that remain unflagged).

* or tagging

There are 2 cases depending on whether the sequence $\{n_k : k \geq 1\}$ terminates or not.

Case (A) : it terminates.

(unflagged)

In this case, there must plainly exist some entry x_M such that $x_n = x_M$ for infinitely many n . Let the list of such $n > M$ be called \mathcal{S} . For $n \in \mathcal{S}$, we have

$$\left\{ \begin{array}{l} x_n = x_M, \quad |y_n - x_M| = |y_n - x_n| \leq \frac{B-A}{2^n} \\ \text{hence } y_n \rightarrow x_M \text{ as } n \in \mathcal{S} \text{ approaches } \infty \end{array} \right\}.$$

But $f(x)$ is continuous at x_M . By continuity, once $n \in \mathbb{N}$ is sufficiently large, we must have $|f(y_n) - f(x_M)| < \frac{\varepsilon_0}{100}$. This is a contradiction since, by assumption, for $n \in \mathbb{N}$,

$$\varepsilon_0 \leq |f(y_n) - f(x_n)| \approx |f(y_n) - f(x_M)| < \frac{\varepsilon_0}{100}.$$

Case (A) is therefore impossible!

Case (B): the sequence $n_1 < n_2 < \dots$ approaches ∞ .

Let \mathbb{S} now refer to the (infinite) sequence $\{n_k\}_{k=1}^{\infty}$. By def of the unflagged indices, the numbers x_n are distinct as n ranges thru \mathbb{S} . The set $E = \{x_n : n \in \mathbb{S}\}$ is thus infinite and lies within $[A, B]$.[⊕] By the B-W property, E necessarily has at least one accumulation point ξ and $\xi \in [A, B]$.

We now use the def of accumulation point.

Thus, for each tiny $\delta > 0$, the punctured nbhd $\{0 < |x - \xi| < \delta\}$ must contain at least one (and, hence, infinitely many) x_n with $n \in \mathbb{S}$.

[⊕] Note that $\{x_n : n \in \mathbb{S}\} = \{x_n : n \geq 1\}$ as subsets of $[A, B]$.

(4)

Take $\delta = \frac{B-A}{10}$. Find some $m_1 \in S$, $m_1 \geq n_1$, so that $0 < |x_{m_1} - \xi| < \frac{B-A}{10}$.

Take $\delta = \frac{B-A}{10^2}$. Find some $m_2 \in S$, $m_2 \geq n_2$, so that $0 < |x_{m_2} - \xi| < \frac{B-A}{10^2}$.

Repeat. Repeat. Get $m_\ell \in S$, $m_\ell \geq n_\ell$, and

$$(\star) \quad 0 < |x_{m_\ell} - \xi| < \frac{B-A}{10^\ell} \quad (\text{all } \ell \geq 1).$$

Remember here that $|y_m - x_m| \leq \frac{B-A}{2^m}$ and $|f(y_m) - f(x_m)| \geq \varepsilon_0$ for every $m \in S$.

Since $m_\ell \geq n_\ell$, plainly $m_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. Also $x_{m_\ell} \rightarrow \xi$ by (\star) . Hence, $y_{m_\ell} \rightarrow \xi$ too. But, $f(x)$ is continuous at ξ . (Remember that $\xi \in [A, B]$.)

Accordingly: $f(x_{m_\ell}) \rightarrow f(\xi)$, $f(y_{m_\ell}) \rightarrow f(\xi)$ as $\ell \rightarrow \infty$. Thus: $|f(x_{m_\ell}) - f(y_{m_\ell})| < \frac{\varepsilon_0}{50}$ once ℓ is large enough. But, by assumption, we have

$$\varepsilon_0 \leq |f(x_{m_\ell}) - f(y_{m_\ell})| \quad (\text{since } m_\ell \in S).$$

Case (B) is thus impossible.

Since neither (A) nor (B) can ever hold, there

is no "bad" ε_0 and the THM follows. \blacksquare (5)

The THM on page ① is readily generalized to functions $f(x)$ which are defined and continuous only on some closed bounded set $Q \subseteq \mathbb{R}^1$. One starts by selecting $R > 0$ so that $Q \subseteq [-R, R]$.

The THM can also be generalized to situations wherein Q is a closed bounded set in \mathbb{R}^N . (See footnote on p. ①.)